

Stringy Unification of Type IIA and IIB Supergravities under $\mathcal{N} = 2$ $D = 10$ Supersymmetric Double Field Theory

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To the full order in fermions, we construct $D = 10$ type II supersymmetric double field theory. We spell the precise $\mathcal{N} = 2$ supersymmetry transformation rules as for 32 supercharges. In terms of stringy differential geometry we have previously developed beyond Riemann, our action treats R-R sector democratically and unifies type IIA and IIB supergravities in a manifestly covariant fashion, with respect to $\mathbf{O}(10, 10)$ T-duality and a pair of local Lorentz groups, $\mathbf{Spin}(1, 9) \times \mathbf{Spin}(9, 1)$, in addition to the usual general covariance of the supergravities.

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Strings perceive spacetime in a different way than particles do through Riemannian geometry. While the fundamental object in Riemannian geometry is the metric, string theory puts the Kalb-Ramond B -field and a scalar dilaton on an equal footing along with the metric, since they form a multiplet of T-duality [1–3], a genuine stringy property which is not present in ordinary particle theory.

Although type IIA and IIB supergravities in ten dimensions provide low energy effective descriptions of closed superstrings, once formulated within the Riemannian setup, they do not manifest T-duality or explain the appearance of enhanced symmetries after dimensional reductions [4, 5]. Riemannian geometry appears unable to capture the full stringy structure and string theory may be urging us to look for a novel mathematical framework.

Double Field Theory (DFT) is an alternative yet equivalent description of the closed string massless sector, designed to manifest the $\mathbf{O}(D, D)$ T-duality initially for the NS-NS sector [6–9] (*c.f.* related precedents [10–13]). It formally doubles the spacetime dimension, from D to $D + D$, by introducing additionally T-dual coordinates for the winding mode. Yet, DFT is not truly doubled since it is subject to so called *strong constraint* or *section condition* that all the fields must live on a D -dimensional null hyperplane: the $\mathbf{O}(D, D)$ invariant d’Alembertian operator is trivial acting on arbitrary fields as well as their products,

$$\partial_A \partial^A = \mathcal{J}^{AB} \partial_A \partial_B \simeq 0, \quad \mathcal{J}^{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (1)$$

DFT also unifies the B -field gauge symmetry and the diffeomorphism, as both are generated by generalized Lie derivative [14, 15],

$$\hat{\mathcal{L}}_X T_{A_1 \dots A_n} := X^B \partial_B T_{A_1 \dots A_n} + \omega_T \partial_B X^B T_{A_1 \dots A_n} + \sum_{i=1}^n (\partial_{A_i} X_B - \partial_B X_{A_i}) T_{A_1 \dots A_{i-1} B A_{i+1} \dots A_n}. \quad (2)$$

Since this differs from the ordinary Lie derivative, the underlying differential geometry of DFT departs from Riemannian one and ought to be *stringy*, such as generalized geometry [15–20] or as proposed in [21–30].

In this Letter, as the conclusive work of our own geometric approaches [21–26], we construct, to the full order in fermions, type II (*i.e.* $\mathcal{N} = 2$, 32 supercharges) $D = 10$ supersymmetric double field theory (SDFT), which treats the R-R sector democratically [31–33], and unifies the type IIA and IIB supergravities in a manifestly covariant fashion, with respect to $\mathbf{O}(10, 10)$ T-duality, a pair of local Lorentz groups, $\mathbf{Spin}(1, 9) \times \mathbf{Spin}(9, 1)$, and the DFT-diffeomorphism generated by the generalized Lie derivative, $\hat{\mathcal{L}}_X$ (2).

While we convey the essence of our formalism from [21–26] shortly and Table I summarizes our index gymnastics, for a detailed review and the precise conventions used in this Letter, we refer the reader to the Appendix of Ref.[26].

Index	Representation	Raising & Lowering Indices
A, B, \dots	$\mathbf{O}(10, 10)$ & $\hat{\mathcal{L}}_X$ vector	\mathcal{J}_{AB}
p, q, \dots	$\mathbf{Spin}(1, 9)$ vector	$\eta_{pq} = \text{diag}(- + + \dots +)$
α, β, \dots	$\mathbf{Spin}(1, 9)$ spinor	$C_{+\alpha\beta}, (\gamma^p)^T = C_+ \gamma^p C_+^{-1}$
\bar{p}, \bar{q}, \dots	$\mathbf{Spin}(9, 1)$ vector	$\bar{\eta}_{\bar{p}\bar{q}} = \text{diag}(+ - - \dots -)$
$\bar{\alpha}, \bar{\beta}, \dots$	$\mathbf{Spin}(9, 1)$ spinor	$\bar{C}_{+\bar{\alpha}\bar{\beta}}, (\bar{\gamma}^{\bar{p}})^T = \bar{C}_+ \bar{\gamma}^{\bar{p}} \bar{C}_+^{-1}$

TABLE I. Index for each symmetry representation and the corresponding “metric” to raise or lower the positions.

The field contents of type II SDFT are

$$d, V_{Ap}, \bar{V}_{A\bar{p}}, C^\alpha_{\bar{\alpha}}, \psi^\alpha_{\bar{p}}, \psi^{\bar{\alpha}}_{\bar{p}}, \rho^\alpha, \rho'^{\bar{\alpha}}. \quad (3)$$

The DFT-dilaton, d , gives rise to a scalar density with weight one, e^{-2d} . The double-vielbeins, $V_{Ap}, \bar{V}_{A\bar{p}}$, satisfy the defining properties:

$$\begin{aligned} V_{Ap} V^A_q &= \eta_{pq}, & \bar{V}_{A\bar{p}} \bar{V}^A_{\bar{q}} &= \bar{\eta}_{\bar{p}\bar{q}}, \\ V_{Ap} \bar{V}^A_{\bar{q}} &= 0, & V_{Ap} V_B^p + \bar{V}_{A\bar{p}} \bar{V}_B^{\bar{p}} &= \mathcal{J}_{AB}, \end{aligned} \quad (4)$$

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such that they generate a pair of orthogonal projections, $P_{AB} = V_A^p V_{Bp}$, $\bar{P}_{AB} = \bar{V}_A^{\bar{p}} \bar{V}_{B\bar{p}}$, satisfying

$$\begin{aligned} P_A^B P_B^C &= P_A^C, & \bar{P}_A^B \bar{P}_B^C &= \bar{P}_A^C, & P_A^B \bar{P}_B^C &= 0, \\ P_A^B + \bar{P}_A^B &= \delta_A^B, & P_A^B V_{Bp} &= V_{Ap}, & \bar{P}_A^B \bar{V}_{B\bar{p}} &= \bar{V}_{A\bar{p}}, \\ \bar{P}_A^B V_{Bp} &= 0, & P_A^B \bar{V}_{B\bar{p}} &= 0. \end{aligned} \quad (5)$$

The R-R potential, $\mathcal{C}^{\alpha}_{\bar{\alpha}}$, is in the bi-fundamental spinorial representation of the local Lorentz group, $\mathbf{Spin}(1, 9) \times \mathbf{Spin}(9, 1)$, and carries chirality,

$$\gamma^{(11)} \mathcal{C} \bar{\gamma}^{(11)} = \pm \mathcal{C}. \quad (6)$$

Hereafter the upper sign is for type IIA and the lower sign is for type IIB. The gravitinos and the DFT-dilatinos are Majorana-Weyl spinors satisfying $\gamma^{(11)} \psi_{\bar{p}} = +\psi_{\bar{p}}$, $\bar{\gamma}^{(11)} \psi'_p = \pm \psi'_p$, $\gamma^{(11)} \rho = -\rho$, $\bar{\gamma}^{(11)} \rho' = \mp \rho'$.

The master semi-covariant derivative [24, 26],

$$\mathcal{D}_A = \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A, \quad (7)$$

contains generically three kinds of connections: Γ_A for the DFT-diffeomorphism (2), Φ_A for $\mathbf{Spin}(1, 9)$, and $\bar{\Phi}_A$ for $\mathbf{Spin}(9, 1)$ local Lorentz symmetries. Contracted with the projections or the double-vielbeins properly, it can produce various fully covariant derivatives.

The master semi-covariant derivative is compatible with all the constants in Table I (“metrics” and gamma matrices). Further, it annihilates the whole NS-NS sector completely, $\mathcal{D}_A d = \mathcal{D}_A V_{Bp} = \mathcal{D}_A \bar{V}_{B\bar{p}} = 0$. The connections are then related to each other through

$$\begin{aligned} \Phi_{Apq} &= V^B_p \nabla_A V_{Bq}, & \bar{\Phi}_{A\bar{p}\bar{q}} &= \bar{V}^B_{\bar{p}} \nabla_A \bar{V}_{B\bar{q}}, \\ \Gamma_{ABC} &= V_B^p D_A V_{Cp} + \bar{V}_B^{\bar{p}} D_A \bar{V}_{C\bar{p}}, \end{aligned} \quad (8)$$

The Lagrangian of type II SDFT we construct in this Letter is

$$\begin{aligned} \mathcal{L}_{\text{Type II}} &= e^{-2d} \left[\frac{1}{8} (P^{AB} P^{CD} - \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD} + \frac{1}{2} \text{Tr}(\mathcal{F} \bar{\mathcal{F}}) - i \bar{\rho} \mathcal{F} \rho' + i \bar{\psi}_{\bar{p}} \gamma_q \mathcal{F} \bar{\gamma}^{\bar{p}} \psi'^q \right. \\ &\quad \left. + i \frac{1}{2} \bar{\rho} \gamma^p \mathcal{D}_p^* \rho - i \bar{\psi}^{\bar{p}} \mathcal{D}_{\bar{p}}^* \rho - i \frac{1}{2} \bar{\psi}^{\bar{p}} \gamma^q \mathcal{D}_{\bar{q}}^* \psi_{\bar{p}} - i \frac{1}{2} \bar{\rho}' \bar{\gamma}^{\bar{p}} \mathcal{D}_{\bar{p}}'^* \rho' + i \bar{\psi}'^{\bar{p}} \mathcal{D}_p'^* \rho' + i \frac{1}{2} \bar{\psi}'^{\bar{p}} \bar{\gamma}^{\bar{q}} \mathcal{D}_{\bar{q}}'^* \psi'_p \right]. \end{aligned} \quad (14)$$

First of all, the curvature, S_{ABCD} , is set by the following specific connection,

$$\begin{aligned} \Gamma_{ABC} &= \Gamma_{ABC}^0 + i \frac{1}{3} \bar{\rho} \gamma_{ABC} \rho - 2i \bar{\rho} \gamma_{BC} \psi_A \\ &\quad - i \frac{1}{3} \bar{\psi}^{\bar{p}} \gamma_{ABC} \psi_{\bar{p}} + 4i \bar{\psi}_B \gamma_A \psi_C + i \frac{1}{3} \bar{\rho}' \bar{\gamma}_{ABC} \rho' \\ &\quad - 2i \bar{\rho}' \bar{\gamma}_{BC} \psi'_A - i \frac{1}{3} \bar{\psi}'^{\bar{p}} \bar{\gamma}_{ABC} \psi'_{\bar{p}} + 4i \bar{\psi}'_B \bar{\gamma}_A \psi'_C. \end{aligned} \quad (15)$$

The master derivatives in the fermionic kinetic terms are twofold: \mathcal{D}_A^* for the unprimed fermions and $\mathcal{D}_A'^*$ for the

where $\nabla_A = \partial_A + \Gamma_A$ and $D_A = \partial_A + \Phi_A + \bar{\Phi}_A$.

In particular, as the DFT analogy of the Riemannian Christoffel connection, the *torsionless connection*, Γ_A^0 , can be uniquely singled out [23, 26] (c.f. [29]):

$$\begin{aligned} \Gamma_{CAB}^0 &= 2(P \partial_C P \bar{P})_{[AB]} + 2(\bar{P}_{[A}^D \bar{P}_{B]}^E - P_{[A}^D P_{B]}^E) \partial_D P_{EC} \\ &\quad - \frac{4}{9} (\bar{P}_{C[A} \bar{P}_{B]}^D + P_{C[A} P_{B]}^D) (\partial_D d + (P \partial^E P \bar{P})_{[ED]}), \end{aligned} \quad (9)$$

such that a generic torsionful connection in DFT assumes the form:

$$\Gamma_{CAB} = \Gamma_{CAB}^0 + \Delta_{Cpq} V_A^p V_B^q + \bar{\Delta}_{C\bar{p}\bar{q}} \bar{V}_A^{\bar{p}} \bar{V}_B^{\bar{q}}, \quad (10)$$

where Δ_{Cpq} , $\bar{\Delta}_{C\bar{p}\bar{q}}$, correspond to torsions, see e.g. (15).

The R-R field strength, $\mathcal{F}^{\alpha}_{\bar{\alpha}}$, is defined by [26]

$$\mathcal{F} := \mathcal{D}_+^0 \mathcal{C}, \quad (11)$$

where \mathcal{D}_+^0 is one of the two fully covariant, *nilpotent* differential operators, \mathcal{D}_{\pm}^0 , which are set by the torsionless connection (9), and may act on an arbitrary $\mathbf{Spin}(1, 9) \times \mathbf{Spin}(9, 1)$ bi-fundamental field, $\mathcal{T}^{\alpha}_{\bar{\beta}}$:

$$\mathcal{D}_{\pm}^0 \mathcal{T} := \gamma^A \mathcal{D}_A^0 \mathcal{T} \pm \gamma^{(11)} \mathcal{D}_A^0 \mathcal{T} \bar{\gamma}^A, \quad (\mathcal{D}_{\pm}^0)^2 \mathcal{T} \simeq 0. \quad (12)$$

The DFT semi-covariant curvature, S_{ABCD} , is defined in terms of the standard, yet non-covariant curvature, R_{ABCD} [23, 25, 26]:

$$\begin{aligned} S_{ABCD} &:= \frac{1}{2} (R_{ABCD} + R_{CDAB} - \Gamma_{AB}^E \Gamma_{ECD}) , \\ R_{CDAB} &= \partial_A \Gamma_{BCD} + \Gamma_{AC}^E \Gamma_{BED} - (A \leftrightarrow B), \end{aligned} \quad (13)$$

and with the help of the projections, it can generate fully covariant curvatures: $(P^{AB} P^{CD} - \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD}$ (scalar) and $V^A_p \bar{V}^B_{\bar{q}} S^C_{ACB} + \frac{1}{2} \mathcal{D}_{\bar{r}} \bar{\Delta}_{p\bar{q}}^{\bar{r}} + \frac{1}{2} \mathcal{D}_r \Delta_{\bar{q}p}^r$ (Ricci-type), where we put $\mathcal{D}_r = V^A_r \mathcal{D}_A$, $\mathcal{D}_{\bar{r}} = \bar{V}^A_{\bar{r}} \mathcal{D}_A$.

primed fermions. They are given by their own connections,

$$\begin{aligned} \Gamma_{ABC}^* &= \Gamma_{ABC} - i \frac{11}{96} \bar{\rho} \gamma_{ABC} \rho + i \frac{5}{4} \bar{\rho} \gamma_{BC} \psi_A \\ &\quad + i \frac{5}{24} \bar{\psi}^{\bar{p}} \gamma_{ABC} \psi_{\bar{p}} - 2i \bar{\psi}_B \gamma_A \psi_C + i \frac{5}{2} \bar{\rho}' \bar{\gamma}_{BC} \psi'_A, \\ \Gamma_{ABC}'^* &= \Gamma_{ABC} - i \frac{11}{96} \bar{\rho}' \bar{\gamma}_{ABC} \rho' + i \frac{5}{4} \bar{\rho}' \bar{\gamma}_{BC} \psi'_A \\ &\quad + i \frac{5}{24} \bar{\psi}'^{\bar{p}} \bar{\gamma}_{ABC} \psi'_{\bar{p}} - 2i \bar{\psi}'_B \bar{\gamma}_A \psi'_C + i \frac{5}{2} \bar{\rho} \gamma_{BC} \psi_A. \end{aligned} \quad (16)$$

With the charge conjugation of the R-R field strength, $\tilde{\mathcal{F}} = \tilde{C}_+^{-1} \mathcal{F}^T C_+$, the trace, $\text{Tr}(\mathcal{F}\tilde{\mathcal{F}})$, is over the $\text{Spin}(1,9)$ spinorial indices. We emphasize that, contracted with the double-vielbeins properly, each term in the Lagrangian (14) is fully covariant.

The type II SDFT Lagrangian (14) is pseudo: an additional *self-duality* relation needs to be imposed by hand on the R-R field strength,

$$\tilde{\mathcal{F}}_- := (1 - \gamma^{(11)}) (\mathcal{F} - i\frac{1}{2}\rho\tilde{\rho}' + i\frac{1}{2}\gamma^p\psi_{\bar{q}}\tilde{\psi}'_p\tilde{\gamma}^{\bar{q}}) \equiv 0. \quad (17)$$

Under arbitrary infinitesimal variations of all the fields, in terms of some shorthand notations,

$$\begin{aligned} \widetilde{\delta\rho} &:= \delta\bar{\rho} - \frac{1}{4}\delta V_{Bq}\bar{\rho}\gamma^{Bq}, \\ \widetilde{\delta\psi^{\bar{p}}} &:= \delta\bar{\psi}^{\bar{p}} - \delta\bar{V}^{B\bar{p}}\bar{\psi}_B - \frac{1}{4}\delta V_{Bq}\bar{\psi}^{\bar{p}}\gamma^{Bq}, \\ \widetilde{\delta\rho'} &:= \delta\bar{\rho}' - \frac{1}{4}\delta\bar{V}_{B\bar{q}}\bar{\rho}'\tilde{\gamma}^{B\bar{q}}, \\ \widetilde{\delta\psi'^p} &:= \delta\bar{\psi}'^p - \delta\bar{V}^{Bp}\bar{\psi}'_B - \frac{1}{4}\delta\bar{V}_{B\bar{q}}\bar{\psi}'^p\tilde{\gamma}^{B\bar{q}}, \\ \widetilde{\delta\mathcal{C}} &:= \delta\mathcal{C} - \mathcal{C}\delta d + \frac{1}{4}\delta V_{Ap}\gamma^A\mathcal{C} - \frac{1}{4}\delta\bar{V}_{A\bar{p}}\mathcal{C}\tilde{\gamma}^{A\bar{p}} \\ &\quad + \frac{1}{2}\delta V_{Ap}\gamma^{(11)}\gamma^p\mathcal{C}\tilde{\gamma}^A, \end{aligned} \quad (18)$$

and a covariant Ricci-type curvature,

$$\begin{aligned} \tilde{S}_{p\bar{q}} &:= V^A{}_p\bar{V}^B{}_{\bar{q}}S^C{}_{ACB} + 2i\bar{\psi}_{\bar{q}}\tilde{\mathcal{D}}_p\rho - i\bar{\psi}^{\bar{p}}\gamma_p\tilde{\mathcal{D}}_{\bar{q}}\psi_{\bar{p}} \\ &\quad + 2i\bar{\psi}'_p\tilde{\mathcal{D}}_{\bar{q}}\rho' - i\bar{\psi}'^q\tilde{\gamma}_{\bar{q}}\tilde{\mathcal{D}}'_p\psi'_q + i\bar{\rho}\gamma_p\tilde{\mathcal{D}}_{\bar{q}}\rho + i\bar{\rho}'\tilde{\gamma}_{\bar{q}}\tilde{\mathcal{D}}'_p\rho', \end{aligned} \quad (19)$$

the Lagrangian transforms, up to total derivatives, as

$$\begin{aligned} \delta\mathcal{L}_{\text{Type II}} &\simeq -2\delta d \times \mathcal{L}_{\text{Type II}} + \delta\Gamma_{ABC} \times 0 \\ &\quad + \frac{1}{2}e^{-2d}\delta V^{Bp}\bar{V}_B{}^{\bar{q}} \left[\tilde{S}_{p\bar{q}} + \text{Tr}(\mathcal{F}\tilde{\gamma}_{\bar{q}}\tilde{\mathcal{F}}\gamma_p) \right] \\ &\quad - ie^{-2d}\widetilde{\delta\psi^{\bar{p}}} \left(\tilde{\mathcal{D}}_{\bar{p}}\rho + \gamma^p\tilde{\mathcal{D}}_p\psi_{\bar{p}} - \gamma^p\mathcal{F}\tilde{\gamma}_{\bar{p}}\psi'_p \right) \\ &\quad + ie^{-2d}\widetilde{\delta\rho} \left(\gamma^p\tilde{\mathcal{D}}_p\rho - \tilde{\mathcal{D}}_{\bar{p}}\psi^{\bar{p}} - \mathcal{F}\rho' \right) \\ &\quad + ie^{-2d}\widetilde{\delta\psi'^p} \left(\tilde{\mathcal{D}}'_p\rho' + \tilde{\gamma}^{\bar{p}}\tilde{\mathcal{D}}_{\bar{p}}\psi'_p - \tilde{\gamma}^{\bar{p}}\mathcal{F}\gamma_p\psi_{\bar{p}} \right) \\ &\quad - ie^{-2d}\widetilde{\delta\rho'} \left(\tilde{\gamma}^{\bar{p}}\tilde{\mathcal{D}}_{\bar{p}}\rho' - \tilde{\mathcal{D}}'_p\psi'^p - \tilde{\mathcal{F}}\rho \right) \\ &\quad + e^{-2d}\text{Tr} \left[\tilde{\mathcal{F}}_- \left(\delta d\tilde{\mathcal{F}} - \frac{1}{2}\delta V^A{}_p\bar{V}_A{}^{\bar{q}}\tilde{\gamma}_{\bar{q}}\tilde{\mathcal{F}}\gamma_p \right) - \mathcal{D}^0_- \tilde{\mathcal{F}}_- \widetilde{\delta\mathcal{C}} \right], \end{aligned} \quad (20)$$

where we further set for $\tilde{\mathcal{D}}_A, \tilde{\mathcal{D}}'_A$,

$$\begin{aligned} \tilde{\Gamma}_{ABC} &= \Gamma_{ABC} - i\frac{23}{54}\bar{\rho}\gamma_{ABC}\rho + i\frac{23}{27}\bar{\rho}\gamma_{BC}\psi_A \\ &\quad + i\frac{23}{54}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} - i\frac{73}{18}\bar{\psi}_B\gamma_A\psi_C - i\frac{5}{4}\bar{\rho}'\tilde{\gamma}_{ABC}\rho' \\ &\quad + i\frac{5}{2}\bar{\rho}'\tilde{\gamma}_{BC}\psi'_A + i\frac{5}{4}\bar{\psi}'^p\tilde{\gamma}_{ABC}\psi'_p - 5i\bar{\psi}'_B\tilde{\gamma}_A\psi'_C, \\ \tilde{\Gamma}'_{ABC} &= \Gamma_{ABC} - i\frac{23}{54}\bar{\rho}'\tilde{\gamma}_{ABC}\rho' + i\frac{23}{27}\bar{\rho}'\tilde{\gamma}_{BC}\psi'_A \\ &\quad + i\frac{23}{54}\bar{\psi}'^p\tilde{\gamma}_{ABC}\psi'_p - i\frac{73}{18}\bar{\psi}'_B\tilde{\gamma}_A\psi'_C - i\frac{5}{4}\bar{\rho}\gamma_{ABC}\rho \\ &\quad + i\frac{5}{2}\bar{\rho}\gamma_{BC}\psi_A + i\frac{5}{4}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} - 5i\bar{\psi}_B\gamma_A\psi_C. \end{aligned} \quad (21)$$

Each line in (20) then gives rise to an equation of motion. In particular, the on-shell Lagrangian vanishes, and the DFT-generalization of the Einstein equation reads

$$\tilde{S}_{p\bar{q}} + \text{Tr}(\mathcal{F}\tilde{\gamma}_{\bar{q}}\tilde{\mathcal{F}}\gamma_p) = 0. \quad (22)$$

The self-duality (17) implies the equation of motion for the R-R potential, $\mathcal{D}^0_- \tilde{\mathcal{F}}_- = 0$. As in $\mathcal{N} = 1$ SDFT [25], the 1.5 formalism, ‘ $\delta\Gamma_{ABC} \times 0$ ’, fixes the connection, Γ_{ABC} , as (15).

The $\mathcal{N} = 2$ supersymmetry transformation rules are

$$\begin{aligned} \delta_\varepsilon d &= -i\frac{1}{2}(\bar{\varepsilon}\rho + \bar{\varepsilon}'\rho'), \\ \delta_\varepsilon V_{Ap} &= i\bar{V}_A{}^{\bar{q}}(\bar{\varepsilon}'\tilde{\gamma}_{\bar{q}}\psi'_p - \bar{\varepsilon}\gamma_p\psi_{\bar{q}}), \\ \delta_\varepsilon \bar{V}_{A\bar{p}} &= iV_A{}^q(\bar{\varepsilon}\gamma_q\psi_{\bar{p}} - \bar{\varepsilon}'\tilde{\gamma}_{\bar{p}}\psi'_q), \\ \delta_\varepsilon \mathcal{C} &= i\frac{1}{2}(\gamma^p\bar{\varepsilon}\tilde{\psi}'_p - \bar{\varepsilon}\rho' - \psi_{\bar{p}}\bar{\varepsilon}'\tilde{\gamma}^{\bar{p}} + \rho\bar{\varepsilon}') \\ &\quad + \mathcal{C}\delta_\varepsilon d - \frac{1}{2}(\bar{V}_A{}^{\bar{q}}\delta_\varepsilon V_{Ap})\gamma^{(d+1)}\gamma^p\mathcal{C}\tilde{\gamma}^{\bar{q}}, \\ \delta_\varepsilon \rho &= -\gamma^p\hat{\mathcal{D}}_p\varepsilon + i\frac{1}{2}\gamma^p\varepsilon\tilde{\psi}'_p\rho' - i\gamma^p\psi_{\bar{q}}\bar{\varepsilon}'\tilde{\gamma}_{\bar{q}}\psi'_p, \\ \delta_\varepsilon \rho' &= -\tilde{\gamma}^{\bar{p}}\hat{\mathcal{D}}_{\bar{p}}\varepsilon' + i\frac{1}{2}\tilde{\gamma}^{\bar{p}}\varepsilon'\tilde{\psi}_{\bar{p}}\rho - i\tilde{\gamma}^{\bar{q}}\psi'_p\bar{\varepsilon}\tilde{\gamma}^p\psi_{\bar{q}}, \\ \delta_\varepsilon \psi_{\bar{p}} &= \hat{\mathcal{D}}_{\bar{p}}\varepsilon + (\mathcal{F} - i\frac{1}{2}\gamma^q\rho\tilde{\psi}'_q + i\frac{1}{2}\psi^{\bar{q}}\rho'\tilde{\gamma}_{\bar{q}})\tilde{\gamma}_{\bar{p}}\varepsilon' \\ &\quad + i\frac{1}{4}\varepsilon\tilde{\psi}_{\bar{p}}\rho + i\frac{1}{2}\psi_{\bar{p}}\bar{\varepsilon}\rho, \\ \delta_\varepsilon \psi'_p &= \hat{\mathcal{D}}'_p\varepsilon' + (\tilde{\mathcal{F}} - i\frac{1}{2}\tilde{\gamma}^{\bar{q}}\rho'\tilde{\psi}_{\bar{q}} + i\frac{1}{2}\psi'^q\bar{\rho}\tilde{\gamma}_{\bar{q}})\gamma_p\varepsilon \\ &\quad + i\frac{1}{4}\varepsilon'\tilde{\psi}'_p\rho' + i\frac{1}{2}\psi'_p\bar{\varepsilon}'\rho', \end{aligned} \quad (23)$$

where $\varepsilon, \varepsilon'$ are Majorana-Weyl $\mathcal{N} = 2$ supersymmetry parameters with chiralities, $\gamma^{(11)}\varepsilon = \varepsilon$, $\tilde{\gamma}^{(11)}\varepsilon' = \pm\varepsilon'$, and $\hat{\mathcal{D}}_A, \hat{\mathcal{D}}'_A$ are given by

$$\begin{aligned} \hat{\Gamma}_{ABC} &= \Gamma_{ABC} - i\frac{17}{48}\bar{\rho}\gamma_{ABC}\rho + i\frac{5}{2}\bar{\rho}\gamma_{BC}\psi_A \\ &\quad + i\frac{1}{4}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} - 3i\bar{\psi}'_B\tilde{\gamma}_A\psi'_C, \\ \hat{\Gamma}'_{ABC} &= \Gamma_{ABC} - i\frac{17}{48}\bar{\rho}'\tilde{\gamma}_{ABC}\rho' + i\frac{5}{2}\bar{\rho}'\tilde{\gamma}_{BC}\psi'_A \\ &\quad + i\frac{1}{4}\bar{\psi}'^p\tilde{\gamma}_{ABC}\psi'_p - 3i\bar{\psi}'_B\tilde{\gamma}_A\psi'_C. \end{aligned} \quad (24)$$

Under the $\mathcal{N} = 2$ supersymmetry (23), the Lagrangian transforms, disregarding total derivatives, as

$$\delta_\varepsilon \mathcal{L}_{\text{Type II}} \simeq -\frac{1}{8}e^{-2d}\bar{V}_A{}^{\bar{q}}\delta_\varepsilon V_{Ap}\text{Tr} \left(\gamma^p\tilde{\mathcal{F}}_- \tilde{\gamma}^{\bar{q}}\tilde{\mathcal{F}}_- \right). \quad (25)$$

This verifies, to the full order in fermions, the supersymmetric invariance of the type II SDFT action modulo the self-duality (17). For consistency, the supersymmetric variation of the self-dual relation (17) is closed by the equations of motion of the gravitinos,

$$\begin{aligned} \delta_\varepsilon \tilde{\mathcal{F}}_- &= -i \left(\tilde{\mathcal{D}}_{\bar{p}}\rho + \gamma^p\tilde{\mathcal{D}}_p\psi_{\bar{p}} - \gamma^p\mathcal{F}\tilde{\gamma}_{\bar{p}}\psi'_p \right) \bar{\varepsilon}'\tilde{\gamma}^{\bar{p}} \\ &\quad - i\gamma^p\varepsilon \left(\tilde{\mathcal{D}}'_p\rho' + \tilde{\mathcal{D}}'_p\psi'_p\tilde{\gamma}^{\bar{p}} - \tilde{\psi}_{\bar{p}}\gamma_p\mathcal{F}\tilde{\gamma}^{\bar{p}} \right). \end{aligned} \quad (26)$$

Double-vielbeins admit parametrization [23, 26]:

$$V_{Ap} = \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})_p{}^\mu \\ (B+e)_{\nu p} \end{pmatrix}, \quad \bar{V}_{A\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{e}^{-1})_{\bar{p}}{}^\mu \\ (B+\bar{e})_{\nu\bar{p}} \end{pmatrix}, \quad (27)$$

where $e_\mu{}^p, \bar{e}_\nu{}^{\bar{p}}$ are two copies of ten-dimensional vielbeins corresponding to the same spacetime metric,

and $B_{\mu\nu}$ is the Kalb-Ramond two-form gauge field, with $B_{\mu p} = B_{\mu\nu}(e^{-1})_p{}^\nu$ and $B_{\mu\bar{p}} = B_{\mu\nu}(\bar{e}^{-1})_{\bar{p}}{}^\nu$. After turning off the winding mode coordinate dependency as a specific solution to the section condition (1), the parametrization (27) renders SDFT to the generalized geometry reformulation of supergravity [19, 20]. Further, as discussed in [26], equating the two vielbeins, $e_\mu{}^p \equiv \bar{e}_\mu{}^{\bar{p}}$, breaks the double local Lorentz groups, $\mathbf{Spin}(1, 9) \times \mathbf{Spin}(9, 1)$, to its diagonal subgroup and truncates type II SDFT to the ‘democratic supergravity’ [33]. Hence, our work provides the $\mathbf{O}(10, 10)$ manifest DFT-generalization as well as the full order supersymmetric completion of Refs.[19, 20, 33].

The diagonal gauge fixing also modifies the $\mathbf{O}(10, 10)$ transformation rule to call for a compensating local Lorentz rotation [26], such that fermions and R-R sector are no longer $\mathbf{O}(10, 10)$ singlet, and $\mathbf{O}(10, 10)$ T-duality group may flip the chirality of the theory, resulting in

the exchange of type IIA and IIB supergravities. In fact [26], the R-R sector can be mapped to an $\mathbf{O}(10, 10)$ spinor [34–37].

The uplift of type II SDFT to \mathcal{M} -theory, or the extension of $\mathbf{O}(10, 10)$ T-duality to E_{11} U-duality, remains as a challenging future work, *c.f.* [38–47]. Setting the primed fermions and the R-R sector trivial consistently truncates the above $\mathcal{N} = 2$ SDFT to the previously constructed $\mathcal{N} = 1$ SDFT [25] (up to a change of convention for the charge conjugation [26]).

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APPENDICES

Appendices are structured as follows.

- Appendix A: The variation of the Lagrangian induced by arbitrary transformations of all the fields, Eq.(20).
- Appendix B: The verification of the $\mathcal{N} = 2$ supersymmetric invariance of the type II SDFT action, Eq.(25).
- Appendix C: Relevant Fierz identities.

Appendix A: Variation of the Lagrangian under arbitrary transformations of fields

It is worth while to note that, under arbitrary variations of all the fields, the following identities, ‘ \simeq ’, hold up to total derivatives and the section constraint (1).

For the double-vielbein, generic (torsionful) connection and curvature,

$$\begin{aligned}\delta V_{Ap} &= \bar{P}_A{}^B \delta V_{Bp} + V_A{}^q \delta V_{B[p} V_{q]}^B, & \delta \bar{V}_{A\bar{p}} &= P_A{}^B \delta \bar{V}_{B\bar{p}} + \bar{V}_A{}^{\bar{q}} \delta \bar{V}_{B[\bar{p}} \bar{V}_{\bar{q}]}^B, \\ \delta \Phi_{Apq} &= \mathcal{D}_A (V^B{}_p \delta V_{Bq}) + V^B{}_p V^C{}_q \delta \Gamma_{ABC}, & \delta \bar{\Phi}_{A\bar{p}\bar{q}} &= \mathcal{D}_A (\bar{V}^B{}_{\bar{p}} \delta \bar{V}_{B\bar{q}}) + \bar{V}^B{}_{\bar{p}} \bar{V}^C{}_{\bar{q}} \delta \Gamma_{ABC}, \\ \delta S_{ABCD} &= \mathcal{D}_{[A} \delta \Gamma_{B]CD} + \mathcal{D}_{[C} \delta \Gamma_{D]AB} - \frac{3}{2} \Gamma_{[ABE]} \delta \Gamma^E{}_{CD} - \frac{3}{2} \Gamma_{[CDE]} \delta \Gamma^E{}_{AB}.\end{aligned}\tag{A1}$$

Further with the fermions,

$$\begin{aligned}\delta V_{Ap} \bar{\psi}_{\bar{p}} \gamma^{Ap} \gamma^{abc} \psi^{\bar{p}} \bar{\rho} \gamma_{abc} \rho + \delta V_{Ap} \bar{\rho} \gamma^{Ap} \gamma^{abc} \rho \bar{\psi}_{\bar{p}} \gamma_{abc} \psi^{\bar{p}} &= 0, \\ \delta \bar{V}_{A\bar{p}} \bar{\psi}'_p \bar{\gamma}^{A\bar{p}} \bar{\gamma}^{\bar{a}\bar{b}\bar{c}} \psi'^p \bar{\rho}' \bar{\gamma}_{\bar{a}\bar{b}\bar{c}} \rho' + \delta \bar{V}_{A\bar{p}} \bar{\rho}' \bar{\gamma}^{A\bar{p}} \bar{\gamma}^{\bar{a}\bar{b}\bar{c}} \rho' \bar{\psi}'_p \bar{\gamma}_{\bar{a}\bar{b}\bar{c}} \psi'^p &= 0.\end{aligned}\tag{A2}$$

For the NS-NS sector of the Lagrangian,

$$\delta \left[\frac{1}{8} (P^{AB} P^{CD} - \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD} \right] \simeq \frac{1}{2} \delta V^{Ap} \bar{V}_A{}^{\bar{q}} S_{p\bar{q}} - \frac{3}{8} \delta \Gamma_{ABC} (P^B{}_D P^C{}_E - \bar{P}^B{}_D \bar{P}^C{}_E) \Gamma^{[ADE]}.\tag{A3}$$

For an arbitrary bi-fundamental quantity, $\mathcal{M}^{\alpha}{}_{\bar{\alpha}}$, with the charge conjugation, $\bar{\mathcal{M}} = \bar{C}_+^{-1} \mathcal{M}^T C_+$,

$$\begin{aligned}e^{-2d} \text{Tr}(\delta \mathcal{F} \bar{\mathcal{M}}) &\simeq e^{-2d} \delta d \text{Tr}(\mathcal{F} \bar{\mathcal{M}}) \\ &+ e^{-2d} \text{Tr} \left[\left(-\delta \mathcal{C} + \mathcal{C} \delta d - \frac{1}{4} \delta V_{Ap} \gamma^{Ap} \mathcal{C} - \frac{1}{2} \bar{V}^A{}_{\bar{p}} \delta V_{Aq} \gamma^{(11)} \gamma^q \mathcal{C} \bar{\gamma}^{\bar{p}} + \frac{1}{4} \delta \bar{V}_{A\bar{p}} \mathcal{C} \bar{\gamma}^{A\bar{p}} \right) \overline{\mathcal{D}}_- \bar{\mathcal{M}} \right. \\ &\quad \left. + \left(-\frac{1}{4} \delta V_{Ap} \gamma^{Ap} \mathcal{F} - \frac{1}{2} \bar{V}^A{}_{\bar{p}} \delta V_{Aq} \gamma^{(11)} \gamma^q \mathcal{F} \bar{\gamma}^{\bar{p}} + \frac{1}{4} \delta \bar{V}_{A\bar{p}} \mathcal{F} \bar{\gamma}^{A\bar{p}} \right) \bar{\mathcal{M}} \right].\end{aligned}\tag{A4}$$

Hence, for the R-R sector of the Lagrangian, we obtain

$$\begin{aligned}&\delta \left[e^{-2d} \left(\frac{1}{2} \mathcal{F}^{\alpha\bar{\alpha}} \mathcal{F}_{\alpha\bar{\alpha}} - i \bar{\rho} \mathcal{F} \rho' + i \bar{\psi}_{\bar{p}} \gamma_q \mathcal{F} \bar{\gamma}^{\bar{p}} \psi'^q \right) \right] \\ &\simeq e^{-2d} \delta d \left(i \bar{\rho} \mathcal{F} \rho' - i \bar{\psi}_{\bar{p}} \gamma_q \mathcal{F} \bar{\gamma}^{\bar{p}} \psi'^q \right) \\ &\quad + e^{-2d} \left[-i \left(\delta \bar{\rho} - \frac{1}{4} \delta V_{Bq} \bar{\rho} \gamma^{Bq} \right) \mathcal{F} \rho' + i \left(\delta \bar{\psi}_{\bar{p}} - \delta \bar{V}^B{}_{\bar{p}} \bar{\psi}_B - \frac{1}{4} \delta V_{Bq} \bar{\psi}_{\bar{p}} \gamma^{Bq} \right) \gamma^q \mathcal{F} \bar{\gamma}^{\bar{p}} \psi'_q \right] \\ &\quad + e^{-2d} \left[+i \left(\delta \bar{\rho}' - \frac{1}{4} \delta \bar{V}_{B\bar{q}} \bar{\rho}' \bar{\gamma}^{B\bar{q}} \right) \bar{\mathcal{F}} \rho - i \left(\delta \bar{\psi}'_p - \delta V^B{}_p \bar{\psi}'_B - \frac{1}{4} \delta \bar{V}_{B\bar{q}} \bar{\psi}'_p \bar{\gamma}^{B\bar{q}} \right) \bar{\gamma}^{\bar{q}} \bar{\mathcal{F}} \gamma^p \psi_{\bar{q}} \right] \\ &\quad + \frac{1}{2} e^{-2d} \delta V^{Ap} \bar{V}_A{}^{\bar{q}} \text{Tr} \left[\gamma^{(11)} \left(\mathcal{F} - i \rho \rho' + i \gamma^r \psi_{\bar{s}} \bar{\psi}'_r \bar{\gamma}^{\bar{s}} \right) \bar{\gamma}_{\bar{q}} \bar{\mathcal{F}} \gamma_p \right] \\ &\quad - e^{-2d} \left(\delta \mathcal{C} - \mathcal{C} \delta d + \frac{1}{4} \delta V_{Ap} \gamma^{Ap} \mathcal{C} + \frac{1}{2} \delta V_{Ap} \gamma^{(11)} \gamma^p \mathcal{C} \bar{\gamma}^A - \frac{1}{4} \delta \bar{V}_{A\bar{p}} \mathcal{C} \bar{\gamma}^{A\bar{p}} \right)^{\alpha\bar{\alpha}} \\ &\quad \times \left[\mathcal{D}_-^0 \left(\mathcal{F} - i \rho \rho' + i \gamma^r \psi_{\bar{s}} \bar{\psi}'_r \bar{\gamma}^{\bar{s}} \right) \right]_{\alpha\bar{\alpha}}.\end{aligned}\tag{A5}$$

For the fermionic kinetic terms, from (A1), we have

$$\begin{aligned}
& e^{-2d} \delta \left(i \frac{1}{2} \bar{\rho} \gamma^p \mathcal{D}_p^* \rho - i \bar{\psi}^{\bar{p}} \mathcal{D}_{\bar{p}}^* \rho - i \frac{1}{2} \bar{\psi}^{\bar{p}} \gamma^q \mathcal{D}_q^* \psi_{\bar{p}} - i \frac{1}{2} \bar{\rho}' \bar{\gamma}^{\bar{p}} \mathcal{D}_{\bar{p}}'^* \rho' + i \bar{\psi}'^p \mathcal{D}_p'^* \rho' + i \frac{1}{2} \bar{\psi}'^p \bar{\gamma}^{\bar{q}} \mathcal{D}_{\bar{q}}'^* \psi_p' \right) \\
& \simeq i \frac{1}{2} e^{-2d} \delta V^{Bp} \bar{V}_{B\bar{q}} \left(\bar{\rho} \gamma_p \mathcal{D}_{\bar{q}}^* \rho + 2 \bar{\psi}_{\bar{q}} \mathcal{D}_p^* \rho - \bar{\psi}^{\bar{p}} \gamma_p \mathcal{D}_{\bar{q}}^* \psi_{\bar{p}} + \bar{\rho}' \bar{\gamma}_{\bar{q}} \mathcal{D}_p'^* \rho' + 2 \bar{\psi}_p' \mathcal{D}_{\bar{q}}'^* \rho' - \bar{\psi}'^q \bar{\gamma}_{\bar{q}} \mathcal{D}_p'^* \psi_q' \right) \\
& + i e^{-2d} \left(\delta \bar{\rho} - \frac{1}{4} \delta V_{Bq} \bar{\rho} \gamma^{Bq} \right) \left(\gamma^p \mathcal{D}_p^* \rho - \mathcal{D}_{\bar{p}}^* \psi^{\bar{p}} \right) \\
& - i e^{-2d} \left(\delta \bar{\psi}^{\bar{p}} - \delta \bar{V}^{B\bar{p}} \bar{\psi}_B - \frac{1}{4} \delta V_{Bq} \bar{\psi}^{\bar{p}} \gamma^{Bq} \right) \left(\mathcal{D}_{\bar{p}}^* \rho + \gamma^p \mathcal{D}_p^* \psi_{\bar{p}} \right) \\
& - i e^{-2d} \left(\delta \bar{\rho}' - \frac{1}{4} \delta \bar{V}_{B\bar{q}} \bar{\rho}' \bar{\gamma}^{B\bar{q}} \right) \left(\bar{\gamma}^{\bar{p}} \mathcal{D}_{\bar{p}}'^* \rho' - \mathcal{D}_p'^* \psi'^p \right) \\
& + i e^{-2d} \left(\delta \bar{\psi}'^p - \delta V^{Bp} \bar{\psi}'_B - \frac{1}{4} \delta \bar{V}_{B\bar{q}} \bar{\psi}'^p \bar{\gamma}^{B\bar{q}} \right) \left(\mathcal{D}_p'^* \rho' + \bar{\gamma}^{\bar{p}} \mathcal{D}_{\bar{p}}'^* \psi_p' \right) \\
& + i e^{-2d} \delta \Gamma_{ABC}^* \left(\frac{1}{8} \bar{\rho} \gamma^{ABC} \rho - \frac{1}{4} \bar{\psi}^A \gamma^{BC} \rho - \frac{1}{8} \bar{\psi}^{\bar{p}} \gamma^{ABC} \psi_{\bar{p}} - \frac{1}{2} \bar{\psi}^B \gamma^A \psi^C \right) \\
& - i e^{-2d} \delta \Gamma_{ABC}'^* \left(\frac{1}{8} \bar{\rho}' \bar{\gamma}^{ABC} \rho' - \frac{1}{4} \bar{\psi}'^A \bar{\gamma}^{BC} \rho' - \frac{1}{8} \bar{\psi}'^p \bar{\gamma}^{ABC} \psi_p' - \frac{1}{2} \bar{\psi}'^B \bar{\gamma}^A \psi'^C \right).
\end{aligned} \tag{A6}$$

Here we let the connections assume the following generic forms:

$$\begin{aligned}
\Gamma_{ABC}^* &= \Gamma_{ABC} + a_1 \bar{\rho} \gamma_{ABC} \rho + a_2 \bar{\rho} \gamma_{BC} \psi_A + a_3 \bar{\psi}_{\bar{p}} \gamma_{ABC} \psi^{\bar{p}} + a_4 \bar{\psi}_B \gamma_A \psi_C \\
&+ a_1' \bar{\rho}' \bar{\gamma}_{ABC} \rho' + a_2' \bar{\rho}' \bar{\gamma}_{BC} \psi_A' + a_3' \bar{\psi}'_{\bar{p}} \bar{\gamma}_{ABC} \psi'^p + a_4' \bar{\psi}'_B \bar{\gamma}_A \psi_C', \\
\Gamma_{ABC}'^* &= \Gamma_{ABC} + a_1 \bar{\rho}' \bar{\gamma}_{ABC} \rho' + a_2 \bar{\rho}' \bar{\gamma}_{BC} \psi_A' + a_3 \bar{\psi}'_{\bar{p}} \bar{\gamma}_{ABC} \psi'^p + a_4 \bar{\psi}'_B \bar{\gamma}_A \psi_C' \\
&+ a_1' \bar{\rho} \gamma_{ABC} \rho + a_2' \bar{\rho} \gamma_{BC} \psi_A + a_3' \bar{\psi}_{\bar{p}} \gamma_{ABC} \psi^{\bar{p}} + a_4' \bar{\psi}_B \gamma_A \psi_C.
\end{aligned} \tag{A7}$$

It is easy to check that, a_1' and a_3' decouple from the fermionic kinetic terms (A6), and only the linear combination, $a_2' - \frac{1}{2} a_4'$ alone is relevant among the four primed coefficients, $\{a_1', a_2', a_3', a_4'\}$. Without loss of generality, henceforth we put

$$a_1' = a_3' = 0. \tag{A8}$$

We proceed to compute the variations of Γ_{ABC}^* and $\Gamma_{ABC}'^*$ (A7), for which we first note

$$\begin{aligned}
& \delta \left(a_1' \bar{\rho}' \bar{\gamma}_{ABC} \rho' + a_2' \bar{\rho}' \bar{\gamma}_{BC} \psi_A' + a_3' \bar{\psi}'_{\bar{p}} \bar{\gamma}_{ABC} \psi'^p + a_4' \bar{\psi}'_B \bar{\gamma}_A \psi_C' \right) \times \left(\frac{1}{8} \bar{\rho} \gamma^{ABC} \rho - \frac{1}{4} \bar{\psi}^A \gamma^{BC} \rho - \frac{1}{8} \bar{\psi}^{\bar{p}} \gamma^{ABC} \psi_{\bar{p}} - \frac{1}{2} \bar{\psi}^B \gamma^A \psi^C \right) \\
& = \delta V^{Ap} \bar{V}_{A\bar{p}} \left(\frac{1}{2} a_1' \bar{\rho}' \bar{\gamma}_{\bar{p}\bar{q}\bar{r}} \rho' \bar{\psi}^{\bar{q}} \gamma_p \psi^{\bar{r}} + \frac{1}{2} a_3' \bar{\psi}'_{\bar{q}} \bar{\gamma}_{\bar{p}\bar{q}\bar{r}} \psi'^q \bar{\psi}^{\bar{q}} \gamma_p \psi^{\bar{r}} - \frac{1}{8} a_4' \bar{\psi}'^q \bar{\gamma}_{\bar{p}} \psi'^r \bar{\rho} \gamma_{pqr} \rho + \frac{1}{8} a_4' \bar{\psi}'^q \bar{\gamma}_{\bar{p}} \psi'^r \bar{\psi}_{\bar{q}} \gamma_{pqr} \psi^{\bar{q}} \right) \\
& - \frac{1}{2} a_2' \left(\delta \bar{\rho}' - \frac{1}{4} \delta \bar{V}_{B\bar{p}} \bar{\rho}' \bar{\gamma}^{B\bar{p}} \right) \bar{\gamma}^{\bar{q}\bar{r}} \psi'^p \bar{\psi}_{\bar{q}} \gamma_p \psi_{\bar{r}} - \frac{1}{2} a_2' \left(\delta \bar{\psi}'^p - \delta V^{Bp} \bar{\psi}'_B - \frac{1}{4} \delta \bar{V}_{B\bar{q}} \bar{\psi}'^p \bar{\gamma}^{B\bar{q}} \right) \bar{\gamma}^{\bar{r}\bar{s}} \rho' \bar{\psi}_{\bar{r}} \gamma_p \psi_{\bar{s}} \\
& - \frac{1}{2} a_4' \left(\delta \bar{\psi}'^p - \delta V^{Bp} \bar{\psi}'_B - \frac{1}{4} \delta \bar{V}_{B\bar{q}} \bar{\psi}'^p \bar{\gamma}^{B\bar{q}} \right) \bar{\gamma}^{\bar{r}} \psi'^q \bar{\rho} \gamma_{pq} \psi_{\bar{r}}.
\end{aligned} \tag{A9}$$

Yet, with (A8) taken, we just need

$$\begin{aligned}
& \delta(a_1 \bar{\rho} \gamma_{ABC} \rho + a_2 \bar{\rho} \gamma_{BC} \psi_A + a_3 \bar{\psi}_B \gamma_{ABC} \psi^{\bar{B}} + a_4 \bar{\psi}_B \gamma_A \psi_C + a'_2 \bar{\rho}' \gamma_{BC} \psi'_A + a'_4 \bar{\psi}'_B \gamma_A \psi'_C) \\
& \quad \times \left(\frac{1}{8} \bar{\rho} \gamma^{ABC} \rho - \frac{1}{4} \bar{\psi}^A \gamma^{BC} \rho - \frac{1}{8} \bar{\psi}^{\bar{B}} \gamma^{ABC} \psi_{\bar{B}} - \frac{1}{2} \bar{\psi}^B \gamma^A \psi^C \right) \\
& = \delta V^{Ap} \bar{V}_A^{\bar{p}} \left[-\left(\frac{1}{4} a_1 + \frac{1}{8} a_2\right) \bar{\rho} \gamma^{st} \psi_{\bar{p}} \bar{\rho} \gamma_{pst} \rho + \left(\frac{1}{8} a_2 - \frac{1}{4} a_3\right) \bar{\rho} \gamma^{st} \psi_{\bar{p}} \bar{\psi}_{\bar{q}} \gamma_{pst} \psi^{\bar{q}} \right. \\
& \quad \left. - \frac{1}{8} a'_4 \bar{\psi}'^q \gamma_{\bar{p}} \psi'^r \bar{\rho} \gamma_{pqr} \rho + \frac{1}{8} a'_4 \bar{\psi}'^q \gamma_{\bar{p}} \psi'^r \bar{\psi}_{\bar{q}} \gamma_{pqr} \psi^{\bar{q}} \right] \\
& \quad + \left(\delta \bar{\rho} - \frac{1}{4} \delta V_{Bp} \bar{\rho} \gamma^{Bp} \right) \left[\frac{1}{4} \gamma_{rst} \rho \left(-a_1 + \frac{1}{16} a_2 \right) \bar{\psi}_{\bar{p}} \gamma^{rst} \psi^{\bar{p}} \right] \\
& \quad + \left(\delta \bar{\psi}^{\bar{p}} - \delta \bar{V}^{B\bar{p}} \bar{\psi}_B - \frac{1}{4} \delta V_{Bp} \bar{\psi}^{\bar{p}} \gamma^{Bp} \right) \left[\frac{1}{4} \gamma_{rst} \psi_{\bar{p}} \left(\left(\frac{1}{16} a_2 + a_3 \right) \bar{\rho} \gamma^{rst} \rho - \left(a_3 + \frac{1}{6} a_4 \right) \bar{\psi}_{\bar{q}} \gamma^{rst} \psi^{\bar{q}} \right) \right] \\
& \quad - \frac{1}{2} a'_2 \left(\delta \bar{\rho}' - \frac{1}{4} \delta \bar{V}_{B\bar{p}} \bar{\rho}' \gamma^{B\bar{p}} \right) \bar{\gamma}^{\bar{q}\bar{r}} \psi'^p \bar{\psi}_{\bar{q}} \gamma_p \psi_{\bar{r}} \\
& \quad - \frac{1}{2} a'_2 \left(\delta \bar{\psi}'^p - \delta V^{Bp} \bar{\psi}'_B - \frac{1}{4} \delta \bar{V}_{B\bar{q}} \bar{\psi}'^p \gamma^{B\bar{q}} \right) \bar{\gamma}^{\bar{r}\bar{s}} \rho' \bar{\psi}_{\bar{r}} \gamma_p \psi_{\bar{s}} \\
& \quad - \frac{1}{2} a'_4 \left(\delta \bar{\psi}'^p - \delta V^{Bp} \bar{\psi}'_B - \frac{1}{4} \delta \bar{V}_{B\bar{q}} \bar{\psi}'^p \gamma^{B\bar{q}} \right) \bar{\gamma}^{\bar{p}} \psi'^q \bar{\rho} \gamma_{pq} \psi_{\bar{p}},
\end{aligned} \tag{A10}$$

and similarly

$$\begin{aligned}
& \delta(a_1 \bar{\rho}' \gamma_{ABC} \rho' + a_2 \bar{\rho}' \gamma_{BC} \psi'_A + a_3 \bar{\psi}'_p \gamma_{ABC} \psi'^p + a_4 \bar{\psi}'_B \gamma_A \psi'_C + a'_2 \bar{\rho}' \gamma_{BC} \psi_A + a'_4 \bar{\psi}'_B \gamma_A \psi_C) \\
& \quad \times \left(\frac{1}{8} \bar{\rho}' \gamma^{ABC} \rho' - \frac{1}{4} \bar{\psi}'^A \gamma^{BC} \rho' - \frac{1}{8} \bar{\psi}'^{\bar{p}} \gamma^{ABC} \psi_p - \frac{1}{2} \bar{\psi}'^B \gamma^A \psi'^C \right) \\
& = -\delta V^{Ap} \bar{V}_A^{\bar{p}} \left[-\left(\frac{1}{4} a_1 + \frac{1}{8} a_2\right) \bar{\rho}' \gamma^{\bar{s}\bar{t}} \psi'_p \bar{\rho}' \gamma_{\bar{p}\bar{s}\bar{t}} \rho' + \left(\frac{1}{8} a_2 - \frac{1}{4} a_3\right) \bar{\rho}' \gamma^{\bar{s}\bar{t}} \psi'_p \bar{\psi}'_{\bar{q}} \gamma_{\bar{p}\bar{s}\bar{t}} \psi'^q \right. \\
& \quad \left. - \frac{1}{8} a'_4 \bar{\psi}'^{\bar{q}} \gamma_p \psi'^{\bar{r}} \bar{\rho}' \gamma_{\bar{p}\bar{q}\bar{r}} \rho' + \frac{1}{8} a'_4 \bar{\psi}'^{\bar{q}} \gamma_p \psi'^{\bar{r}} \bar{\psi}'_{\bar{q}} \gamma_{\bar{p}\bar{q}\bar{r}} \psi'^q \right] \\
& \quad + \left(\delta \bar{\rho}' - \frac{1}{4} \delta \bar{V}_{B\bar{p}} \bar{\rho}' \gamma^{B\bar{p}} \right) \left[\frac{1}{4} \bar{\gamma}_{\bar{r}\bar{s}\bar{t}} \rho' \left(-a_1 + \frac{1}{16} a_2 \right) \bar{\psi}'_p \bar{\gamma}^{\bar{r}\bar{s}\bar{t}} \psi'^p \right] \\
& \quad + \left(\delta \bar{\psi}'^p - \delta V^{Bp} \bar{\psi}'_B - \frac{1}{4} \delta \bar{V}_{B\bar{q}} \bar{\psi}'^p \gamma^{B\bar{q}} \right) \left[\frac{1}{4} \bar{\gamma}_{\bar{r}\bar{s}\bar{t}} \psi'_p \left(\left(\frac{1}{16} a_2 + a_3 \right) \bar{\rho}' \gamma^{\bar{r}\bar{s}\bar{t}} \rho' - \left(a_3 + \frac{1}{6} a_4 \right) \bar{\psi}'_{\bar{q}} \gamma^{\bar{r}\bar{s}\bar{t}} \psi'^q \right) \right] \\
& \quad - \frac{1}{2} a'_2 \left(\delta \bar{\rho}' - \frac{1}{4} \delta V_{Bp} \bar{\rho}' \gamma^{Bp} \right) \gamma^{qr} \psi_{\bar{p}} \bar{\psi}'_q \bar{\gamma}_{\bar{p}} \psi'_r \\
& \quad - \frac{1}{2} a'_2 \left(\delta \bar{\psi}'^{\bar{p}} - \delta \bar{V}^{B\bar{p}} \bar{\psi}_B - \frac{1}{4} \delta V_{Bp} \bar{\psi}^{\bar{p}} \gamma^{Bp} \right) \gamma^{qr} \rho \bar{\psi}'_q \bar{\gamma}_{\bar{p}} \psi'_r \\
& \quad - \frac{1}{2} a'_4 \left(\delta \bar{\psi}'^{\bar{p}} - \delta \bar{V}^{B\bar{p}} \bar{\psi}_B - \frac{1}{4} \delta V_{Bp} \bar{\psi}^{\bar{p}} \gamma^{Bp} \right) \gamma^q \psi_{\bar{q}} \bar{\rho}' \gamma_{\bar{p}\bar{q}} \psi'_q.
\end{aligned} \tag{A11}$$

The variation of the fermionic kinetic terms (A6) now assumes the desired expression:

$$\begin{aligned}
& e^{-2d} \delta \left(i \frac{1}{2} \bar{\rho} \gamma^p \mathcal{D}_p^* \rho - i \bar{\psi}^{\bar{p}} \mathcal{D}_{\bar{p}}^* \rho - i \frac{1}{2} \bar{\psi}^{\bar{p}} \gamma^q \mathcal{D}_q^* \psi_{\bar{p}} - i \frac{1}{2} \bar{\rho}' \gamma^{\bar{p}} \mathcal{D}_{\bar{p}}'^* \rho' + i \bar{\psi}'^p \mathcal{D}_p'^* \rho' + i \frac{1}{2} \bar{\psi}'^p \gamma^{\bar{q}} \mathcal{D}_{\bar{q}}'^* \psi'_p \right) \\
& \simeq i \frac{1}{2} e^{-2d} \delta V^{Bp} \bar{V}_B^{\bar{q}} \left(\bar{\rho} \gamma_p \mathcal{D}_{\bar{q}}^{\sharp} \rho + 2 \bar{\psi}_{\bar{q}} \tilde{\mathcal{D}}_p \rho - \bar{\psi}^{\bar{p}} \gamma_p \mathcal{D}_{\bar{q}}^b \psi_{\bar{p}} + \bar{\rho}' \gamma_{\bar{q}} \mathcal{D}_{\bar{p}}^{\sharp} \rho' + 2 \bar{\psi}'_p \tilde{\mathcal{D}}_{\bar{q}} \rho' - \bar{\psi}'^q \gamma_{\bar{q}} \mathcal{D}_{\bar{p}}^b \psi'_q \right) \\
& \quad + i e^{-2d} \left(\delta \bar{\rho} - \frac{1}{4} \delta V_{Bq} \bar{\rho} \gamma^{Bq} \right) \left(\gamma^p \mathcal{D}_p^{\sharp} \rho - \tilde{\mathcal{D}}_{\bar{p}} \psi^{\bar{p}} \right) \\
& \quad - i e^{-2d} \left(\delta \bar{\psi}^{\bar{p}} - \delta \bar{V}^{B\bar{p}} \bar{\psi}_B - \frac{1}{4} \delta V_{Bq} \bar{\psi}^{\bar{p}} \gamma^{Bq} \right) \left(\tilde{\mathcal{D}}_{\bar{p}} \rho + \gamma^p \mathcal{D}_p^b \psi_{\bar{p}} \right) \\
& \quad - i e^{-2d} \left(\delta \bar{\rho}' - \frac{1}{4} \delta \bar{V}_{B\bar{q}} \bar{\rho}' \gamma^{B\bar{q}} \right) \left(\bar{\gamma}^{\bar{p}} \mathcal{D}_{\bar{p}}^{\sharp} \rho' - \tilde{\mathcal{D}}'_p \psi'^p \right) \\
& \quad + i e^{-2d} \left(\delta \bar{\psi}'^p - \delta V^{Bp} \bar{\psi}'_B - \frac{1}{4} \delta \bar{V}_{B\bar{q}} \bar{\psi}'^p \gamma^{B\bar{q}} \right) \left(\tilde{\mathcal{D}}'_p \rho' + \bar{\gamma}^{\bar{p}} \mathcal{D}_{\bar{p}}^b \psi'_p \right) \\
& \quad + i e^{-2d} \delta \Gamma_{ABC} \left(\frac{1}{8} \bar{\rho} \gamma^{ABC} \rho - \frac{1}{4} \bar{\psi}^A \gamma^{BC} \rho - \frac{1}{8} \bar{\psi}^{\bar{p}} \gamma^{ABC} \psi_{\bar{p}} - \frac{1}{2} \bar{\psi}^B \gamma^A \psi^C \right. \\
& \quad \left. - \frac{1}{8} \bar{\rho}' \gamma^{ABC} \rho' + \frac{1}{4} \bar{\psi}'^A \gamma^{BC} \rho' + \frac{1}{8} \bar{\psi}'^p \gamma^{ABC} \psi_p + \frac{1}{2} \bar{\psi}'^B \gamma^A \psi'^C \right),
\end{aligned} \tag{A12}$$

and the Lagrangian transforms up to total derivatives as

$$\begin{aligned}
\delta\mathcal{L}_{\text{Type II}} \simeq & -2\delta d \times \mathcal{L}_{\text{Type II}} \\
& +\delta\Gamma_{ABC} \times 0 \\
& +\frac{1}{2}e^{-2d}\delta V^{Bp}\bar{V}_B^{\bar{q}} \left[\tilde{S}_{p\bar{q}} + \text{Tr}(\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}}\gamma_p) \right] \\
& -ie^{-2d} \left(\delta\bar{\psi}^{\bar{p}} - \delta\bar{V}^{B\bar{p}}\bar{\psi}_B - \frac{1}{4}\delta V_{Bq}\bar{\psi}^{\bar{p}}\gamma^{Bq} \right) \left(\tilde{\mathcal{D}}_{\bar{p}}\rho + \gamma^p\mathcal{D}_p^b\psi_{\bar{p}} - \gamma^p\mathcal{F}\bar{\gamma}_{\bar{p}}\psi'_p \right) \\
& +ie^{-2d} \left(\delta\bar{\rho} - \frac{1}{4}\delta V_{Bq}\bar{\rho}\gamma^{Bq} \right) \left(\gamma^p\mathcal{D}_p^\sharp\rho - \tilde{\mathcal{D}}_{\bar{p}}\psi^{\bar{p}} - \mathcal{F}\rho' \right) \\
& +ie^{-2d} \left(\delta\bar{\psi}'^p - \delta V^{Bp}\bar{\psi}'_B - \frac{1}{4}\delta\bar{V}_{B\bar{q}}\bar{\psi}'^p\bar{\gamma}^{B\bar{q}} \right) \left(\tilde{\mathcal{D}}'_p\rho' + \bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}^b\psi'_p - \bar{\gamma}^{\bar{p}}\bar{\mathcal{F}}\gamma_p\psi_{\bar{p}} \right) \\
& -ie^{-2d} \left(\delta\bar{\rho}' - \frac{1}{4}\delta\bar{V}_{B\bar{q}}\bar{\rho}'\bar{\gamma}^{B\bar{q}} \right) \left(\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}^\sharp\rho' - \tilde{\mathcal{D}}'_p\psi'^p - \bar{\mathcal{F}}\rho \right) \\
& +e^{-2d}\text{Tr} \left[\tilde{\mathcal{F}}_- \left(\delta d\bar{\mathcal{F}} - \frac{1}{2}\delta V^{Ap}\bar{V}_A^{\bar{q}}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}}\gamma_p \right) - \mathcal{D}_-^0\tilde{\mathcal{F}}_-\widetilde{\delta\mathcal{C}} \right].
\end{aligned} \tag{A13}$$

Here we set generically

$$\tilde{S}_{p\bar{q}} := S_{p\bar{q}} + 2i\bar{\psi}_{\bar{q}}\tilde{\mathcal{D}}_p\rho - i\bar{\psi}^{\bar{p}}\gamma_p\mathcal{D}_{\bar{q}}^b\psi_{\bar{p}} + 2i\bar{\psi}'_p\tilde{\mathcal{D}}'_{\bar{q}}\rho' - i\bar{\psi}'^q\bar{\gamma}_{\bar{q}}\mathcal{D}_p^b\psi'_q + i\bar{\rho}\gamma_p\mathcal{D}_{\bar{q}}^\sharp\rho + i\bar{\rho}'\bar{\gamma}_{\bar{q}}\mathcal{D}_p^\sharp\rho', \tag{A14}$$

and

$$\begin{aligned}
\Gamma_{ABC}^\sharp &= \Gamma_{ABC}^\star + b_1\bar{\rho}\gamma_{ABC}\rho + b_2\bar{\rho}\gamma_{BC}\psi_A + b_3\bar{\psi}_{\bar{p}}\gamma_{ABC}\psi^{\bar{p}} + b_4\bar{\psi}_B\gamma_A\psi_C \\
&\quad + b'_1\bar{\rho}'\bar{\gamma}_{ABC}\rho' + b'_2\bar{\rho}'\bar{\gamma}_{BC}\psi'_A + b'_3\bar{\psi}'_p\bar{\gamma}_{ABC}\psi'^p + b'_4\bar{\psi}'_B\bar{\gamma}_A\psi'_C, \\
\Gamma_{ABC}^b &= \Gamma_{ABC}^\star + c_1\bar{\rho}\gamma_{ABC}\rho + c_2\bar{\rho}\gamma_{BC}\psi_A + c_3\bar{\psi}_{\bar{p}}\gamma_{ABC}\psi^{\bar{p}} + c_4\bar{\psi}_B\gamma_A\psi_C \\
&\quad + c'_1\bar{\rho}'\bar{\gamma}_{ABC}\rho' + c'_2\bar{\rho}'\bar{\gamma}_{BC}\psi'_A + c'_3\bar{\psi}'_p\bar{\gamma}_{ABC}\psi'^p + c'_4\bar{\psi}'_B\bar{\gamma}_A\psi'_C, \\
\tilde{\Gamma}_{ABC} &= \Gamma_{ABC}^\star + d_1\bar{\rho}\gamma_{ABC}\rho + d_2\bar{\rho}\gamma_{BC}\psi_A + d_3\bar{\psi}_{\bar{p}}\gamma_{ABC}\psi^{\bar{p}} + d_4\bar{\psi}_B\gamma_A\psi_C \\
&\quad + d'_1\bar{\rho}'\bar{\gamma}_{ABC}\rho' + d'_2\bar{\rho}'\bar{\gamma}_{BC}\psi'_A + d'_3\bar{\psi}'_p\bar{\gamma}_{ABC}\psi'^p + d'_4\bar{\psi}'_B\bar{\gamma}_A\psi'_C,
\end{aligned} \tag{A15}$$

$$\begin{aligned}
\Gamma_{ABC}^{\prime\sharp} &= \Gamma_{ABC}^{\prime\star} + b_1\bar{\rho}'\bar{\gamma}_{ABC}\rho' + b_2\bar{\rho}'\bar{\gamma}_{BC}\psi'_A + b_3\bar{\psi}'_p\bar{\gamma}_{ABC}\psi'^p + b_4\bar{\psi}'_B\bar{\gamma}_A\psi'_C \\
&\quad + b'_1\bar{\rho}\gamma_{ABC}\rho + b'_2\bar{\rho}\gamma_{BC}\psi_A + b'_3\bar{\psi}_{\bar{p}}\gamma_{ABC}\psi^{\bar{p}} + b'_4\bar{\psi}_B\gamma_A\psi_C, \\
\Gamma_{ABC}^{\prime b} &= \Gamma_{ABC}^{\prime\star} + c_1\bar{\rho}'\bar{\gamma}_{ABC}\rho' + c_2\bar{\rho}'\bar{\gamma}_{BC}\psi'_A + c_3\bar{\psi}'_p\bar{\gamma}_{ABC}\psi'^p + c_4\bar{\psi}'_B\bar{\gamma}_A\psi'_C \\
&\quad + c'_1\bar{\rho}\gamma_{ABC}\rho + c'_2\bar{\rho}\gamma_{BC}\psi_A + c'_3\bar{\psi}_{\bar{p}}\gamma_{ABC}\psi^{\bar{p}} + c'_4\bar{\psi}_B\gamma_A\psi_C, \\
\tilde{\Gamma}_{ABC}' &= \Gamma_{ABC}^{\prime\star} + d_1\bar{\rho}'\bar{\gamma}_{ABC}\rho' + d_2\bar{\rho}'\bar{\gamma}_{BC}\psi'_A + d_3\bar{\psi}'_p\bar{\gamma}_{ABC}\psi'^p + d_4\bar{\psi}'_B\bar{\gamma}_A\psi'_C \\
&\quad + d'_1\bar{\rho}\gamma_{ABC}\rho + d'_2\bar{\rho}\gamma_{BC}\psi_A + d'_3\bar{\psi}_{\bar{p}}\gamma_{ABC}\psi^{\bar{p}} + d'_4\bar{\psi}_B\gamma_A\psi_C,
\end{aligned} \tag{A16}$$

of which the coefficients must satisfy the following nine constraints,

$$\begin{aligned}
a'_4 + b'_4 &= 4c'_1, & a'_2 + c'_2 &= a'_2 - \frac{1}{2}a'_4, \\
a'_4 + c'_4 &= -4c'_3, & a'_4 + d'_4 &= -2(a'_2 - \frac{1}{2}a'_4), \\
a_1 + b_3 &= \frac{1}{16}(a_2 - d_2), & a_3 + c_1 &= -\frac{1}{16}(a_2 - d_2), \\
a_3 - c_3 &= \frac{1}{6}(c_4 - a_4), & a_1 + d_1 &= -\frac{1}{2}(a_2 + b_2), \\
a_3 + d_3 &= \frac{1}{2}(a_2 + c_2).
\end{aligned} \tag{A17}$$

A particularly simple solution is given by

$$\begin{aligned}
b'_1 = c'_1 = d'_1 &= -\frac{1}{2}(a'_2 - \frac{1}{2}a'_4), & b'_2 = c'_2 = d'_2 &= -\frac{1}{2}a'_4, \\
b'_3 = c'_3 = d'_3 &= \frac{1}{2}(a'_2 - \frac{1}{2}a'_4), & b'_4 = c'_4 = d'_4 &= -2a'_2,
\end{aligned} \tag{A18}$$

and

$$\begin{aligned}
b_1 = c_1 = d_1 &= -\frac{1}{9}(a_1 + a_2 + 8a_3), & b_2 = c_2 = d_2 &= -\frac{1}{9}(16a_1 + 7a_2 - 16a_3), \\
b_3 = c_3 = d_3 &= -\frac{1}{9}(8a_1 - a_2 + a_3), & b_4 = c_4 = d_4 &= \frac{1}{3}(16a_1 - 2a_2 + 20a_3 + 3a_4).
\end{aligned} \tag{A19}$$

Specifically, for Γ_{ABC}^* and $\Gamma_{ABC}'^*$ given in (16) as

$$a_1 = -i\frac{11}{96}, \quad a_2 = i\frac{5}{4}, \quad a_3 = i\frac{5}{24}, \quad a_4 = -2i, \quad a'_1 = 0, \quad a'_2 = i\frac{5}{2}, \quad a'_3 = 0, \quad a'_4 = 0, \tag{A20}$$

we achieve (21),

$$\begin{aligned}
\Gamma_{ABC}^\# &= \Gamma_{ABC}^\flat = \tilde{\Gamma}_{ABC} = \Gamma_{ABC} - i\frac{23}{54}\bar{\rho}\gamma_{ABC}\rho + i\frac{23}{27}\bar{\rho}\gamma_{BC}\psi_A + i\frac{23}{54}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} - i\frac{73}{18}\bar{\psi}_B\gamma_A\psi_C \\
&\quad - i\frac{5}{4}\bar{\rho}'\gamma_{ABC}\rho' + i\frac{5}{2}\bar{\rho}'\gamma_{BC}\psi'_A + i\frac{5}{4}\bar{\psi}'^p\gamma_{ABC}\psi'_p - 5i\bar{\psi}'_B\gamma_A\psi'_C, \\
\Gamma_{ABC}'^\# &= \Gamma_{ABC}'^\flat = \tilde{\Gamma}_{ABC}' = \Gamma_{ABC} - i\frac{23}{54}\bar{\rho}'\gamma_{ABC}\rho' + i\frac{23}{27}\bar{\rho}'\gamma_{BC}\psi'_A + i\frac{23}{54}\bar{\psi}'^p\gamma_{ABC}\psi'_p - i\frac{73}{18}\bar{\psi}'_B\gamma_A\psi'_C \\
&\quad - i\frac{5}{4}\bar{\rho}\gamma_{ABC}\rho + i\frac{5}{2}\bar{\rho}\gamma_{BC}\psi_A + i\frac{5}{4}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} - 5i\bar{\psi}_B\gamma_A\psi_C.
\end{aligned} \tag{A21}$$

Alternatively, in a similar fashion to Ref.[25], we might set

$$\begin{aligned}
\tilde{\Gamma}_{ABC} &= \Gamma_{ABC} - i\frac{17}{48}\bar{\rho}\gamma_{ABC}\rho + i\frac{5}{2}\bar{\rho}\gamma_{BC}\psi_A + i\frac{1}{4}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} - 5i\bar{\psi}'_B\gamma_A\psi'_C, \\
\tilde{\Gamma}'_{ABC} &= \Gamma_{ABC} - i\frac{17}{48}\bar{\rho}'\gamma_{ABC}\rho' + i\frac{5}{2}\bar{\rho}'\gamma_{BC}\psi'_A + i\frac{1}{4}\bar{\psi}'^p\gamma_{ABC}\psi'_p - 5i\bar{\psi}_B\gamma_A\psi_C, \\
\Gamma_{ABC}^\# &= \Gamma_{ABC} + i\frac{17}{24}\bar{\rho}\gamma_{BC}\psi_A + i\frac{31}{96}\bar{\psi}^{\bar{r}}\gamma_{ABC}\psi_{\bar{r}}, \\
\Gamma_{ABC}'^\# &= \Gamma_{ABC} + i\frac{17}{24}\bar{\rho}'\gamma_{BC}\psi'_A + i\frac{31}{96}\bar{\psi}'^r\gamma_{ABC}\psi'_r, \\
\Gamma_{ABC}^\flat &= \Gamma_{ABC} - i\frac{31}{96}\bar{\rho}\gamma_{ABC}\rho + i\frac{1}{2}\bar{\rho}\gamma_{BC}\psi_A + i\frac{5}{12}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} - 4i\bar{\psi}_B\gamma_A\psi_C + i\frac{5}{2}\bar{\rho}'\gamma_{BC}\psi'_A, \\
\Gamma_{ABC}'^\flat &= \Gamma_{ABC} - i\frac{31}{96}\bar{\rho}'\gamma_{ABC}\rho' + i\frac{1}{2}\bar{\rho}'\gamma_{BC}\psi'_A + i\frac{5}{12}\bar{\psi}'^p\gamma_{ABC}\psi'_p - 4i\bar{\psi}'_B\gamma_A\psi'_C + i\frac{5}{2}\bar{\rho}\gamma_{BC}\psi_A.
\end{aligned} \tag{A22}$$

That is to say, there are various ways of absorbing the higher order fermionic terms into the torsions as long as the constraint (A17) is satisfied. In this paper, we choose (A21) and hence (21) such that, the entire equations of motion (20) can be written in terms of only two kind of torsions: one for the unprimed fermions and the other for the primed fermions.

Appendix B: $\mathcal{N} = 2$ supersymmetric invariance of the action

Here, we sketch our verification of the $\mathcal{N} = 2$ supersymmetric invariance of the action as in (25), order by order in fermions. We substitute the $\mathcal{N} = 2$ supersymmetry transformation rules (23) into (20) and organize the supersymmetric variation of the Lagrangian as

$$\delta_\varepsilon \mathcal{L}_{\text{Type II}} = \delta_\varepsilon \mathcal{L}_{\text{Type II}}^{[1]} + \delta_\varepsilon \mathcal{L}_{\text{Type II}}^{[3]} + \delta_\varepsilon \mathcal{L}_{\text{Type II}}^{[5]}, \quad (\text{B1})$$

where $\delta_\varepsilon \mathcal{L}_{\text{Type II}}^{[1]}$, $\delta_\varepsilon \mathcal{L}_{\text{Type II}}^{[3]}$ and $\delta_\varepsilon \mathcal{L}_{\text{Type II}}^{[5]}$ denote respectively the linear, cubic and quintic order terms in fermions which are DFT-dilatino and gravitino.

First of all, we focus on the linear order terms which decompose into four parts:

$$\delta_\varepsilon \mathcal{L}_{\text{Type II}}^{[1]} \simeq \Delta_\rho + \Delta_\psi + \Delta_{\mathcal{F}} + \Delta_{\mathcal{F}^2}, \quad (\text{B2})$$

where, disregarding the total derivative terms, we have

$$\begin{aligned} \Delta_\rho &= -ie^{-2d} \left[\frac{1}{8}(\bar{\rho}\varepsilon + \bar{\rho}'\varepsilon')(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD}^0 \right. \\ &\quad \left. + \bar{\rho} \left(\frac{1}{2}\gamma^{pq}[\mathcal{D}_p^0, \mathcal{D}_q^0] + \mathcal{D}^{0A}\mathcal{D}_A^0 \right) \varepsilon - \bar{\rho}' \left(\frac{1}{2}\bar{\gamma}^{\bar{p}\bar{q}}[\mathcal{D}_{\bar{p}}^0, \mathcal{D}_{\bar{q}}^0] + \mathcal{D}^{0A}\mathcal{D}_A^0 \right) \varepsilon' \right], \\ \Delta_\psi &= -ie^{-2d} \left[\frac{1}{2}(\bar{\varepsilon}\gamma^p\psi^{\bar{q}} - \bar{\varepsilon}'\bar{\gamma}^{\bar{q}}\psi'^p)S_{p\bar{q}}^0 + \bar{\psi}^{\bar{q}}\gamma^p[\mathcal{D}_p^0, \mathcal{D}_{\bar{q}}^0]\varepsilon + \bar{\psi}'^p\bar{\gamma}^{\bar{q}}[\mathcal{D}_p^0, \mathcal{D}_{\bar{q}}^0]\varepsilon' \right], \\ \Delta_{\mathcal{F}} &= -ie^{-2d} \left[\bar{\rho}(\mathcal{D}_{\bar{p}}^0\mathcal{F})\bar{\gamma}^{\bar{p}}\varepsilon' - \bar{\rho}'(\mathcal{D}_p^0\bar{\mathcal{F}})\gamma^p\varepsilon + \bar{\psi}_{\bar{p}}(\gamma^p\mathcal{D}_p^0\mathcal{F})\bar{\gamma}^{\bar{p}}\varepsilon' - \bar{\psi}'_p(\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}^0\bar{\mathcal{F}})\gamma^p\varepsilon \right. \\ &\quad \left. + \frac{1}{2}\text{Tr}[(\rho\varepsilon' - \varepsilon\rho' + \gamma^p\varepsilon\bar{\psi}'_p - \bar{\psi}_{\bar{p}}\varepsilon'\bar{\gamma}^{\bar{p}})\overline{\mathcal{D}^0\mathcal{F}}] \right], \\ \Delta_{\mathcal{F}^2} &= +ie^{-2d} \left[\bar{\psi}_{\bar{p}}\gamma_q\mathcal{F}\bar{\gamma}^{\bar{p}}\bar{\mathcal{F}}\gamma^q\varepsilon - \bar{\psi}'_p\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}}\gamma^p\mathcal{F}\bar{\gamma}^{\bar{q}}\varepsilon' + \frac{1}{2}(\bar{\varepsilon}\gamma_p\psi_{\bar{q}} - \bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p)\text{Tr}(\gamma^{(11)}\gamma^p\mathcal{F}\bar{\gamma}^{\bar{q}}\bar{\mathcal{F}}) \right]. \end{aligned} \quad (\text{B3})$$

We show, up to the level matching section constraint (1), each of them vanishes except the last one, $\Delta_{\mathcal{F}^2}$.

1. For Δ_ρ .

We first note

$$\begin{aligned} [\mathcal{D}_A^0, \mathcal{D}_B^0]\varepsilon &= F_{AB}\varepsilon - \Gamma^C{}_{AB}\mathcal{D}_C^0\varepsilon, \quad \mathcal{D}_A^0\mathcal{D}^{0A}\varepsilon = (\partial_A\Phi^{0A} + \Gamma_A{}^{AB}\Phi_B^0 - \Phi_A^0\Phi^{0A})\varepsilon + 2\Phi_A^0\mathcal{D}^{0A}\varepsilon, \\ [\mathcal{D}_A^0, \mathcal{D}_B^0]\varepsilon' &= \bar{F}_{AB}\varepsilon' - \Gamma^C{}_{AB}\mathcal{D}_C^0\varepsilon', \quad \mathcal{D}_A^0\mathcal{D}^{0A}\varepsilon' = (\partial_A\bar{\Phi}^{0A} + \Gamma_A{}^{AB}\bar{\Phi}_B^0 - \bar{\Phi}_A^0\bar{\Phi}^{0A})\varepsilon' + 2\bar{\Phi}_A^0\mathcal{D}^{0A}\varepsilon'. \end{aligned} \quad (\text{B4})$$

Then, due to the identities [26],

$$\begin{aligned} \partial_A\Phi^{0A} + \Phi_A^0\Phi^{0A} + \frac{1}{2}\gamma^{AB}F_{AB} + (\Gamma_B{}^{BA} - \frac{1}{2}\Gamma^A{}_{pq}\gamma^{pq})\Phi_A^0 &\simeq -\frac{1}{4}S_{ABCD}^0P^{AC}P^{BD}, \\ \partial_A\bar{\Phi}^{0A} + \bar{\Phi}_A^0\bar{\Phi}^{0A} + \frac{1}{2}\bar{\gamma}^{AB}\bar{F}_{AB} + (\Gamma_B{}^{BA} - \frac{1}{2}\Gamma^A{}_{\bar{p}\bar{q}}\bar{\gamma}^{\bar{p}\bar{q}})\bar{\Phi}_A^0 &\simeq -\frac{1}{4}S_{ABCD}^0\bar{P}^{AC}\bar{P}^{BD}, \end{aligned} \quad (\text{B5})$$

we obtain (*c.f.* [19, 28])

$$\begin{aligned} \left(\frac{1}{2}\gamma^{pq}[\mathcal{D}_p^0, \mathcal{D}_q^0] + \mathcal{D}^{0A}\mathcal{D}_A^0 \right) \varepsilon &\simeq -\frac{1}{4}P^{AB}P^{CD}S_{ACBD}^0\varepsilon, \\ \left(\frac{1}{2}\bar{\gamma}^{\bar{p}\bar{q}}[\mathcal{D}_{\bar{p}}^0, \mathcal{D}_{\bar{q}}^0] + \mathcal{D}^{0A}\mathcal{D}_A^0 \right) \varepsilon' &\simeq -\frac{1}{4}\bar{P}^{AB}\bar{P}^{CD}S_{ACBD}^0\varepsilon'. \end{aligned} \quad (\text{B6})$$

These simplify Δ_ρ as

$$\Delta_\rho \simeq i\frac{1}{8}e^{-2d}(\bar{\rho}\varepsilon - \bar{\rho}'\varepsilon')(P^{AB}P^{CD} + \bar{P}^{AB}\bar{P}^{CD})S_{ACBD}^0, \quad (\text{B7})$$

and finally from the identity [23, 26],

$$(P^{AB}P^{CD} + \bar{P}^{AB}\bar{P}^{CD})S_{ACBD}^0 \simeq 0, \quad (\text{B8})$$

we note $\Delta_\rho \simeq 0$.

2. For Δ_ψ .
From

$$[\mathcal{D}_p^0, \mathcal{D}_{\bar{q}}^0]\varepsilon \simeq \frac{1}{2}S_{p\bar{q}rs}^0\gamma^{rs}\varepsilon, \quad [\mathcal{D}_p^0, \mathcal{D}_{\bar{q}}^0]\varepsilon' \simeq \frac{1}{2}S_{p\bar{q}\bar{r}\bar{s}}^0\bar{\gamma}^{\bar{r}\bar{s}}\varepsilon', \quad (\text{B9})$$

Δ_ψ reduces to

$$\Delta_\psi \simeq -i\frac{1}{2}e^{-2d}(\bar{\psi}'^p\bar{\gamma}^{\bar{q}}\varepsilon' + \bar{\psi}^{\bar{q}}\gamma^p\varepsilon)(P^{AB} - \bar{P}^{AB})S_{pA\bar{q}B}^0. \quad (\text{B10})$$

Then, from the identity [23, 26],

$$(P^{AB} - \bar{P}^{AB})S_{pA\bar{q}B}^0 \simeq 0, \quad (\text{B11})$$

we verify $\Delta_\psi \simeq 0$.

3. For $\Delta_{\mathcal{F}}$.
Straightforward computation may give

$$\begin{aligned} \Delta_{\mathcal{F}} = & -ie^{-2d}\left[\bar{\rho}(1 - \gamma^{(11)})\mathcal{D}_{\bar{p}}^0\mathcal{F}\bar{\gamma}^{\bar{p}}\varepsilon' + \bar{\varepsilon}(1 + \gamma^{(11)})\gamma^p\mathcal{D}_{\bar{q}}^0\mathcal{F}\bar{\gamma}^{\bar{q}}\psi'_p\right. \\ & \left. - \frac{1}{2}\text{Tr}[(\rho'\bar{\varepsilon} + \psi'_p\bar{\varepsilon}\gamma^p + \varepsilon'\bar{\rho} + \bar{\gamma}^{\bar{p}}\varepsilon'\psi_{\bar{p}})\mathcal{D}_+^0\mathcal{F}]\right]. \end{aligned} \quad (\text{B12})$$

Hence, from the chirality of the fermions and the nilpotent property [26],

$$\mathcal{D}_+^0\mathcal{F} = (\mathcal{D}_+^0)^2\mathcal{C} \simeq 0, \quad (\text{B13})$$

we note $\Delta_{\mathcal{F}} \simeq 0$.

4. For $\Delta_{\mathcal{F}^2}$.

Using the well-known Fierz identities involving a cyclic sum over three spinorial indices (C6), it is easy to show that $\Delta_{\mathcal{F}^2}$ (and hence $\delta_\varepsilon\mathcal{L}_{\text{Type II}}^{[1]}$) reduces to

$$\delta_\varepsilon\mathcal{L}_{\text{Type II}}^{[1]} \simeq \Delta_{\mathcal{F}^2} = i\frac{1}{4}e^{-2d}(\bar{\varepsilon}\gamma_p\psi_{\bar{q}} - \bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p)\text{Tr}[\gamma^p(1 - \gamma^{(11)})\mathcal{F}\bar{\gamma}^{\bar{q}}\bar{\mathcal{F}}]. \quad (\text{B14})$$

We now turn to the higher order terms in fermions. After long and tedious computations, using the various Fierz identities presented in Appendix C, we obtain, for the cubic order terms,

$$\delta_\varepsilon\mathcal{L}_{\text{Type II}}^{[3]} \simeq \frac{1}{2}e^{-2d}(\bar{\varepsilon}\gamma_p\psi_{\bar{q}} - \bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p)(\bar{\rho}\gamma^p\mathcal{F}\bar{\gamma}^{\bar{q}}\rho' - \bar{\psi}_{\bar{s}}\gamma^t\gamma^p\mathcal{F}\bar{\gamma}^{\bar{q}}\bar{\gamma}^{\bar{s}}\psi'_t), \quad (\text{B15})$$

and for the quintic order terms,

$$\delta_\varepsilon\mathcal{L}_{\text{Type II}}^{[5]} = i\frac{1}{4}e^{-2d}(\bar{\varepsilon}\gamma_p\psi_{\bar{q}} - \bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p)\left(\bar{\rho}\gamma^p\gamma^s\bar{\psi}_{\bar{s}}\bar{\psi}'_s\bar{\gamma}^{\bar{s}}\bar{\gamma}^{\bar{q}}\rho' + \frac{1}{2}\bar{\psi}_{\bar{s}}\gamma^s\gamma^p\gamma^t\psi_{\bar{t}}\bar{\psi}'_s\bar{\gamma}^{\bar{s}}\bar{\gamma}^{\bar{q}}\bar{\gamma}^{\bar{t}}\psi'_t\right). \quad (\text{B16})$$

In fact, the cubic order terms decompose into two parts: one involving the R-R field strength, \mathcal{F} , and the other with the torsionless master derivative, \mathcal{D}_A^0 . The former reduces to (B15) and the latter turns out to be a total derivative which we neglect. The computation of the quintic order terms is genuinely algebraic.

At last, adding up (B14), (B15) and (B16), we obtain the final expression (25). This completes our verification of the $\mathcal{N} = 2$ supersymmetric invariance of the action, modulo the self-duality (17), to the full order in fermions.

Appendix C: Fierz identities

With the chiral/anti-chiral projections,

$$\gamma_{\pm} := \frac{1}{2} (1 \pm \gamma^{(11)}) , \quad \bar{\gamma}_{\pm} := \frac{1}{2} (1 \pm \bar{\gamma}^{(11)}) , \quad (C1)$$

where

$$\gamma^{(11)} = \gamma^{012\cdots 9} , \quad \bar{\gamma}^{(11)} = \bar{\gamma}^{012\cdots 9} , \quad (C2)$$

relevant Fierz identities are as follows (*c.f.* [25]).

$$\begin{aligned} (\gamma_-)^{\alpha}{}_{\lambda} (\gamma_+)^{\delta}{}_{\beta} &= \frac{1}{32 \times 5!} (\gamma_{a_5 \cdots a_1} \gamma_-)^{\delta}{}_{\lambda} (\gamma^{a_1 \cdots a_5} \gamma_+)^{\alpha}{}_{\beta} + \sum_{n=1,3} \frac{1}{16 \times n!} (\gamma_{a_n \cdots a_1} \gamma_-)^{\delta}{}_{\lambda} (\gamma^{a_1 \cdots a_n} \gamma_+)^{\alpha}{}_{\beta} , \\ (\gamma_{\pm})^{\alpha}{}_{\lambda} (\gamma_{\pm})^{\delta}{}_{\beta} &= \sum_{n=0,2,4} \frac{1}{16 \times n!} (\gamma_{a_n \cdots a_1} \gamma_{\pm})^{\delta}{}_{\lambda} (\gamma^{a_1 \cdots a_n} \gamma_{\pm})^{\alpha}{}_{\beta} , \end{aligned} \quad (C3)$$

$$\gamma_{a_m \cdots a_1} \gamma^{b_1 \cdots b_n} = \sum_{l=0}^{\min[m,n]} l! \binom{m}{l} \binom{n}{l} \gamma_{[a_m \cdots a_{l+1}} [\gamma^{b_{l+1} \cdots b_n} \delta_{a_1}^{b_1} \cdots \delta_{a_l}^{b_l}] , \quad (C4)$$

$$\begin{aligned} \gamma_p \gamma_{a_1 \cdots a_n} \gamma^p &= (-1)^n (10 - 2n) \gamma_{a_1 \cdots a_n} , \\ \gamma_{pq} \gamma_{a_1 \cdots a_n} \gamma^{pq} &= (-90 + 40n - 4n^2) \gamma_{a_1 \cdots a_n} , \\ \gamma_{pqr} \gamma_{a_1 \cdots a_n} \gamma^{pqr} &= (-1)^n (-720 + 544n - 120n^2 + 8n^3) \gamma_{a_1 \cdots a_n} , \end{aligned} \quad (C5)$$

$$(C_+ \gamma^p \gamma_{\pm})_{(\alpha\beta)} (C_+ \gamma_p \gamma_{\pm})_{\gamma\delta} = 0 , \quad (\bar{C}_+ \bar{\gamma}^{\bar{p}} \bar{\gamma}_{\pm})_{(\bar{\alpha}\bar{\beta})} (\bar{C}_+ \bar{\gamma}_{\bar{p}} \bar{\gamma}_{\pm})_{\bar{\gamma}\bar{\delta}} = 0 . \quad (C6)$$

In particular,

$$\gamma_{abc} \gamma^{stu} = -\gamma^{stu} \gamma_{abc} + 18 \gamma^{[st} \gamma_{[ab} \delta_{c]}^u + 72 \gamma^{[s} \gamma_{[a} \delta_{bc]}^{tu}] - 48 \delta_u^s \delta_b^t \delta_c^u , \quad (C7)$$

$$\gamma^{pq} \gamma_{abc} \gamma^{stu} \gamma_{pq} = -6 \gamma^{stu} \gamma_{abc} + 108 \gamma^{[st} \gamma_{[ab} \delta_{c]}^u - 144 \gamma^{[s} \gamma_{[a} \delta_{bc]}^{tu}] - 864 \delta_u^s \delta_b^t \delta_c^u . \quad (C8)$$

For the chiral gravitinos, $\psi_{\bar{p}} = +\gamma^{(11)} \psi_{\bar{p}}$, and the anti-chiral DFT-dilatinos, $\rho = -\gamma^{(11)} \rho$,

$$\bar{\rho} \gamma^{pqr} \rho (\bar{\rho} \gamma_{pq})_{\alpha} = 0 , \quad \bar{\rho} \gamma^{pqr} \rho (\bar{\rho} \gamma_{pqr})_{\alpha} = 0 , \quad (C9)$$

$$\rho \bar{\rho} = \frac{1}{96} (\bar{\rho} \gamma^{pqr} \rho) \gamma_- \gamma_{pqr} \gamma_+ , \quad \psi^{\bar{p}} \bar{\psi}_{\bar{p}} = \frac{1}{96} (\bar{\psi}^{\bar{p}} \gamma^{pqr} \psi_{\bar{p}}) \gamma_+ \gamma_{pqr} \gamma_- , \quad (C10)$$

$$\frac{1}{16} \bar{\rho} \gamma^{pqr} \rho \bar{\psi}_{\bar{p}} \gamma_{pqr} \psi^{\bar{p}} = -\bar{\rho} \gamma_{pq} \psi_{\bar{p}} \bar{\rho} \gamma^{pq} \psi^{\bar{p}} , \quad (C11)$$

$$\gamma^{pqrst} \varepsilon \bar{\psi}_{\bar{p}} \gamma_{pqrst} = 0 , \quad \gamma^{pqrst} \rho \bar{\psi}_{\bar{p}} \gamma_a \gamma_{pqrst} = 0 , \quad (C12)$$

$$\frac{1}{2}\gamma^q\psi_{\bar{p}}\bar{\varepsilon}\gamma_{pq}\rho - \frac{1}{2}\gamma^q\varepsilon\bar{\psi}_{\bar{p}}\gamma_{pq}\rho + \frac{1}{8}\gamma_p\gamma^{qr}\varepsilon\bar{\psi}_{\bar{p}}\gamma_{qr}\rho + \frac{1}{2}\gamma_p\psi_{\bar{p}}\bar{\varepsilon}\rho + \frac{1}{4}\gamma_p\varepsilon\bar{\psi}_{\bar{p}}\rho + \rho\bar{\varepsilon}\gamma_p\psi_{\bar{p}} = 0, \quad (C13)$$

$$\bar{\psi}_{\bar{s}}\gamma^s\gamma^p\gamma^t\psi_{\bar{t}}\bar{\psi}'_s\bar{\gamma}^{\bar{s}}\bar{\gamma}^{\bar{q}}\bar{\gamma}^{\bar{t}}\psi'_t = \bar{\psi}_{\bar{s}}\gamma^{pst}\psi_{\bar{s}}\bar{\psi}'_s\bar{\gamma}^{\bar{q}}\psi'_t - 2\bar{\psi}_{\bar{q}}\gamma^{pst}\psi_{\bar{s}}\bar{\psi}'_s\bar{\gamma}^{\bar{s}}\psi'_t + \bar{\psi}'_s\bar{\gamma}^{\bar{q}\bar{s}\bar{t}}\psi'_s\bar{\psi}_{\bar{s}}\gamma^p\psi_{\bar{t}} - 2\bar{\psi}'^p\bar{\gamma}^{\bar{q}\bar{s}\bar{t}}\psi'_s\bar{\psi}_{\bar{s}}\gamma^s\psi_{\bar{t}}. \quad (C14)$$

Further, with an additional anti-chiral fermion, $\chi = -\gamma^{(11)}\chi$,

$$\frac{1}{96}\bar{\varepsilon}\gamma^{rst}\psi_{\bar{p}}\bar{\rho}\gamma_{rst}\chi - \frac{3}{2}\bar{\rho}\varepsilon\bar{\psi}_{\bar{p}}\chi - \frac{1}{8}\bar{\varepsilon}\gamma^{rs}\chi\bar{\rho}\gamma_{rs}\psi_{\bar{p}} - \frac{1}{4}\bar{\rho}\psi_{\bar{p}}\bar{\varepsilon}\chi + \frac{1}{2}\bar{\varepsilon}\gamma^r\psi_{\bar{p}}\bar{\rho}\gamma_r\chi = 0. \quad (C15)$$

With the R-R field strength, $\mathcal{F} = \mp\gamma^{(11)}\mathcal{F}\bar{\gamma}^{(11)}$,

$$\begin{aligned} (\gamma_p\psi_{\bar{q}})(\bar{\varepsilon}\gamma^p\mathcal{F}\bar{\gamma}^{\bar{q}}) &= \frac{1}{2}(\bar{\psi}_{\bar{q}}\gamma_p\varepsilon)(\gamma^p\mathcal{F}\bar{\gamma}^{\bar{q}}) - \frac{1}{24}(\bar{\psi}_{\bar{q}}\gamma_{abc}\varepsilon)(\gamma^{abc}\mathcal{F}\bar{\gamma}^{\bar{q}}), \\ (\bar{\varepsilon}\gamma_p\psi_{\bar{q}})(\gamma^p\mathcal{F}\bar{\gamma}^{\bar{q}}) &= -\frac{1}{2}(\gamma^p\varepsilon)(\bar{\psi}_{\bar{q}}\gamma_p\mathcal{F}\bar{\gamma}^{\bar{q}}) + \frac{1}{24}(\gamma^{abc}\varepsilon)(\bar{\psi}_{\bar{q}}\gamma_{abc}\mathcal{F}\bar{\gamma}^{\bar{q}}), \\ (\bar{\varepsilon}\gamma_{abc}\psi_{\bar{q}})(\gamma^{abc}\mathcal{F}\bar{\gamma}^{\bar{q}}) &= -18(\gamma^p\varepsilon)(\bar{\psi}_{\bar{q}}\gamma_p\mathcal{F}\bar{\gamma}^{\bar{q}}) - \frac{1}{2}(\gamma^{abc}\varepsilon)(\bar{\psi}_{\bar{q}}\gamma_{abc}\mathcal{F}\bar{\gamma}^{\bar{q}}), \\ (\gamma^p\mathcal{F}\bar{\gamma}^{\bar{q}}\varepsilon')(\bar{\psi}'_p\bar{\gamma}_{\bar{q}}) &= \frac{1}{2}(\gamma^p\mathcal{F}\bar{\gamma}^{\bar{q}})(\bar{\psi}'_p\bar{\gamma}_{\bar{q}}\varepsilon') - \frac{1}{24}(\gamma^p\mathcal{F}\bar{\gamma}^{\bar{a}\bar{b}\bar{c}})(\bar{\psi}'_p\bar{\gamma}_{\bar{a}\bar{b}\bar{c}}\varepsilon'), \\ (\gamma^p\mathcal{F}\bar{\gamma}^{\bar{q}})(\bar{\psi}'_p\bar{\gamma}_{\bar{q}}\varepsilon') &= -\frac{1}{2}(\gamma^p\mathcal{F}\bar{\gamma}_{\bar{q}}\psi'_p)(\bar{\varepsilon}'\bar{\gamma}^{\bar{q}}) + \frac{1}{24}(\gamma^p\mathcal{F}\bar{\gamma}_{\bar{a}\bar{b}\bar{c}}\psi'_p)(\bar{\varepsilon}'\bar{\gamma}^{\bar{a}\bar{b}\bar{c}}), \\ (\gamma^p\mathcal{F}\bar{\gamma}^{\bar{a}\bar{b}\bar{c}})(\bar{\psi}'_p\bar{\gamma}_{\bar{a}\bar{b}\bar{c}}\varepsilon') &= 18(\gamma^q\mathcal{F}\bar{\gamma}_{\bar{p}}\psi'_q)(\bar{\varepsilon}'\bar{\gamma}^{\bar{p}}) + \frac{1}{2}(\gamma^p\mathcal{F}\bar{\gamma}_{\bar{a}\bar{b}\bar{c}}\psi'_p)(\bar{\varepsilon}'\bar{\gamma}^{\bar{a}\bar{b}\bar{c}}). \end{aligned} \quad (C16)$$
